Macroeconomics II

Assignment 3

1. Solve the following problems, paying particular attention to the appropriate transversality conditions.

(a) Fixed end points

$$\max_u \int_0^1 -u(t)^2 dt\ , \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t)\ , \quad y(0)=1,\ y(1)=0.$$

(b) Constrained end point

$$\max_u \int_0^1 -u(t)^2 dt\ , \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t)\ , y(0)=1, \quad \text{y}(1) \geq 2\ .$$

(c) T free, but $y(T)$ fixed

$$\max_u \int_0^T -1dt\ , \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t)\ , \quad y(0)=5, y(T)=11, \quad u \in [-1,1]$$

2. Characterize the solution to the following investment problem:

$$\max_c \int_o^T e^{-\rho t} U(c(t)) dt$$

subject to $\dot{k}(t) = rk(t) - c(t)$, $k(0)=k_0>0$, $k(T) \geq 0$, and where $\lim_{c \to 0} u^{/}(c) = \infty$, and $u^{//}(c) < 0$.

3. Find the path of x(t) that maximizes

$$V = \int_{t_0}^{\infty} e^{-\rho t} \ln x(t) dt\ ,$$

subject to $\dot{m}(t) = \beta m(t) - x(t)$, $m(t_0) = m_0 > 0$, $\lim_{t \to \infty} m(t) = 0$. Assume that $\rho > \beta$.

4. (The Ramsey-Cass-Koopmans model with a pollution externality). Suppose family utility is given by

$$\int_0^{\infty} \left(U(c(t)) - V(y(t)) \right) e^{-\rho t} dt$$

where U is an increasing strictly concave function and V is an increasing strictly convex function. $V(y(t))$ is the disutility of pollution associated with production.

(a) *Decentralized economy.* The household budget constraint is

$$\dot{k}(t) = r(t)k(t) + w(t) - c(t)\ .$$

Households choose c(t) to maximize lifetime utility, but they ignore the effects of $k(t)$ on $y(t)$. Derive the steady state of the competitive economy.

(b) *Command economy.* The social planner's resource constraint is

$$\dot{k}(t) = f(k(t)) - c(t),$$

and the planner maximizes utility subject to the constraint, taking into account the fact that $y(t) = f(k(t))$. Derive the steady state conditions and compare your answer with part (a).

5. (*The Ramsey-Cass-Koopmans model with leisure*). Consider a competitive economy populated by identical infinitely-lived individuals whose utility functions are given by

$$\int_0^\infty \big(U(c(t)) + V(T - h(t))\big)e^{-\rho t}dt,$$

where U and V are increasing strictly concave functions. $c(t)$ is consumption, T is the number of hours in the day, $h(t)$ is the number of hours spent working, so that $T-h(t)$ is the number of hours of leisure per day. The marginal utilities of consumption and leisure are positive and diminishing. An individual's budget constraint is given by

$$\dot{k}(t) = r(t)k(t) + w(t)h(t) + z(t) - c(t) - \tau w(t)h(t),$$

where $k(t)$ is capital holdings, $r(t)$ is the rental rate on capital, $w(t)$ is the hourly wage, $z(t)$ is a transfer payment from the government, and τ is the tax rate on labor income.

(a) Derive the optimality conditions for the individual.

The per capita net production function for this economy is $y(t) = f(k(t), h(t))$, where f is strictly concave in each input, homogenous of degree one, and the derivatives satisfy $f_{hk} > 0$ and $f_{hh}f_{kk} - (f_{hk})^2 > 0$.

(b) Assume that markets are competitive and that the government maintains a balanced budget. Find and interpret the three conditions that determine steady-state per capita consumption, capital and hours as a function of τ.

(c) Find expressions for the effect of an increase in τ on the steady-state values of c, k, and h. Interpret these results.