THE AREA AND CIRCUMFERENCE OF A CIRCLE

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In grade school we were all given the formulas for the area and circumference of a circle:

$$A = \pi r^2 \text{ and } C = 2\pi r$$

where $\pi \approx 3.14159$. Most likely these formulas were given with no justification, or even an intuitive explanation as to why they are true. In this article I will discuss the two formulas above, I will give elementary proofs of them, and I will discuss their history.

To begin with, how should one interpret the formulas $A = \pi r^2$ and $C = 2\pi r$? It is tempting to take the point of view that first there was this number π "out there," and subsequently it was discovered that π is useful for calculating the area and circumference of a circle. This of course would be misleading. The formulas $A = \pi r^2$ and $C = 2\pi r$ should be interpreted as both a *theorem* in geometry, and the *definition* of π. The theorem says that *there is some constant* k, such that for all circles, the area and circumference of the circle are given by: $A = kr^2$ and $C = 2kr$. The definition says, let us agree to use the symbol π to refer to this constant. To be more precise, as I see it, there are at least seven important ideas associated with the area and circumference formulas given above. I describe these seven ideas below. (In order to understand what follows, it would be helpful to pretend for a moment that you have never heard of the number π. Then I will *define* π for you below.)

Idea 1. The circumference of a circle is directly proportional to the radius of the circle. That is, there is some constant k such that for all circles, $C = kr$. This implies for instance that if you double the radius of a circle, then you double its circumference.

Idea 2. The area of a circle is directly proportional to the square of the radius of the circle. That is, there is some constant h such that for all circles, $A = hr^2$. This implies for instance that if you double the radius of a circle, then you quadruple its area.

Idea 3. The two constants of proportionality mentioned above are related to each other by the equation $k = 2h$.

Notation. Given Idea 3 above, let us agree to use the symbol π to refer to the constant h. Then Idea 1 says that $C = 2\pi r$ and Idea 2 says that $A = \pi r^2$.

Idea 4. The number π is approximately equal to 3. To be more precise, $3\frac{1}{8} < \pi < 3\frac{1}{7}$.

Idea 5. $\pi \approx 3.14159265358979323846$. (I think of this as a separate idea beyond Idea 4 because Idea 4 can actually be discovered by physical experiment, whereas Idea 5 must be discovered by mathematical analysis.)

Idea 6. π is an irrational number. That is π cannot be expressed exactly with any finite number of decimal places, or any fraction of whole numbers.

Idea 7. Not only is π an irrational number, π is actually a *transcendental* number. (I will explain this later.)

Some History

Ideas 1 through 7 above are listed approximately in the order in which they occurred historically. I will give a short description of some of this history.[1]

Ideas 1 and 4 were likely known by prehistoric man, more than 5000 years ago. The reason is that these ideas are fairly easy to discover experimentally. For example, suppose you are at the beach and you are sitting on the wet sand. Suppose also that you have some rope with you. Using a short piece of rope you draw a circle in the wet sand. You do this by holding one end of the rope down with one hand, or a stick. Then you pull the rope taught and revolve it around the center point, while drawing in the wet sand with your finger. If your piece of rope had length r, then you have drawn a circle of radius r. Now you take a second longer piece of rope and you lay the rope down in the circular groove in the sand. You cut the second piece of rope so that it exactly fits once around the circle. Then you compare your short first piece of rope with your longer second piece of rope. By laying off the short piece against the longer piece, you discover that the long piece is a little more than six times as long as the short piece. That is, you discover that the circumference of your circle is a little more than $6r$. If you repeat this experiment several times, you will discover that it always comes out the same way, no matter how big or how small you draw the circle. You have thus discovered that the circumference of a circle is directly proportional to the radius of the circle, with a constant of proportionality a little more than 6. If you measure very carefully you can probably determine that the constant of proportionality is between $6\frac{1}{4}$ and $6\frac{2}{7}$.

It is a little bit more difficult to discover in this manner the relationship between the radius of a circle and its *area*. For this reason, perhaps Idea 2 did not occur as early in history as Idea 1. Nevertheless, it is not very difficult to imagine some simple physical experiments which would allow you to discover the fact that if you double the radius of a circle, you quadruple its area. By carefully estimating the area of a circle (by counting the number of small squares which can be drawn in the circle) it can be discovered that the area of a circle is proportional to the square of the radius, with the constant of proportionality being a little more than three. With very careful measurement it may be discovered that the constant of proportionality is between $3\frac{1}{8}$ and $3\frac{1}{7}$.

At any rate, it is known that Ideas 1, 2, 3, and 4 were known by certain ancient civilizations by about 2000 B.C. In one ancient Egyptian mathematical document called the Rhind Papyrus, the author's process for finding the area of a circle with radius r was to use the formula $A = 4(8/9)^2 r^2$. This amounts to using the value

$$\pi = 4\left(\frac{8}{9}\right)^2 \approx 3.1605.$$

A Babylonian cuneiform tablet discovered at Susa by a French archeological expedition in 1936 asserts that the area of a circle is equal to 2/25 times the square of the circumference. This amounts to using a value of

$$\pi = \frac{25}{8} = 3\frac{1}{8} = 3.125.$$

How did these ancient peoples arrive at their values for π? Nobody knows the answer for certain. From the fact that their values for π are only accurate to the first decimal place, we might guess that some sort of physical estimation was involved, rather than any mathematical analysis. This brings us to the history of Idea 5. Namely, the idea of estimating π accurately to many decimal places. Obviously this kind of accurate estimation can not be done by measuring the relative lengths of two pieces of rope. There are many different ways in which

[1] All of the historical information in this article is taken directly from the two books listed in the references. I will not give any further references for the various historical claims I make in the article.

mathematical analysis can be used to estimate the value of π. I will discuss this a bit later. At this time I would like to focus on history. Historians of science have devoted considerable attention to the attempts throughout history to calculate π with further and further accuracy. (Of course, most of the people who worked on this problem did not use the symbol π. The first use of the symbol π to stand for the ratio of a circle's circumference to its diameter was in 1706 by an obscure English writer, William Jones, in his *Synopsis Palmariorum Matheseos*, or *A New Introduction to the Mathematics.* The reason for the use of π may be that it is the first letter of the Greek word *perimetros* meaning perimeter. In 1748 Leonhard Euler used the symbol π in his famous *Introductio in Analysin Infinitorum.* From this point on the use of the symbol π became universal.)

Without any attempt at completeness, let me just list a few historical records concerning computations of the digits of π. In the third century B.C. the Greek mathematician Archimedes, in his treatise *The Measurement of a Circle*, proved that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

In his work, Archimedes *proved* the validity of Ideas 1, 2, 3 and 4 above. Note that this is different than the ancient Egyptians and Babylonians who probably "knew" of Ideas 1 through 4, but did not prove them. I will talk more about proofs later.

In the first millennium A.D., it was the Chinese who led the way in the approximation of π. In the third century, Liu Hui derived the bounds

$$3.141024 < \pi < 3.142704.$$

And in the fifth century Tsu Chung-Chi obtained

$$3.1415926 < \pi < 3.1415927.$$

Jumping ahead to the sixteenth century, in 1593 Francois Viete set a new record by calculating that

$$3.1415926535 < \pi < 3.1415926537.$$

By the end of the 16th century, π was known to 30 decimal places, by the end of the 18th century it was known to 140 decimal places, and by the end of the 19th century it was known to 526 decimal places. Today, with the use of computers, π can be calculated to many more decimal places. One programmer has found π to $500,000$ decimal places!

But is the programmer who found π to 500,000 places any closer to capturing all of π than the ancient Babylonians who knew that π was a little more than 3? In one sense, the answer is no. This is because π is an irrational number. Its decimal expansion is infinite, and non-repeating. This brings us to the history of Idea 6. Throughout history people were able to find more and more digits of π. Maybe if someone were *very* ambitious and *very* patient, they could find all of the digits of π, or at least find a long pattern of digits which repeated over and over again. Any hope along these lines was dashed in 1767, when the Swiss mathematician Johann Heinrich Lambert proved the irrationality of π. In his treatise *Preliminary Knowledge for Those Who Seek the Quadrature and Rectification of the Circle*, Lambert proved the following theorem: *If x is a rational number other than zero, then* $\tan x$ *cannot be rational.* Since $\tan(\pi/4) = 1$ is rational, $\pi/4$ must be irrational, and so π must be irrational. Some credit for this result must also be given to Adrien-Marie Legendre. In his *Elements of Geometry* (1794) Legendre gave a more rigorous proof of a lemma which Lambert had used in his proof.

Finally, we come to the history of Idea 7, the idea that π is a transcendental number. Look back at the title of Lambert's treatise mentioned in the last paragraph. Lambert refers to the *quadrature* of the circle? What does this mean? This phrase refers to a very old problem in mathematics, the problem of "squaring the circle." In brief the idea is as follows. Suppose you are given a circle drawn on a piece of paper. Can you draw a square whose area is equal to the area of the given circle? But wait, there are some rules! You are allowed to use a writing utensil, a straight edge to draw straight lines, and a compass to draw circles. And that's it. You are not allowed to use a ruler, or a protractor, or any other tools. Furthermore you are not allowed to use the given tools in any way other than the prescribed way. In particular you are not allowed to make any marks on the straight edge in order to record a distance, and you are not allowed to use the legs of the compass in order to transfer a length from one place to another. People worked on the problem of squaring the circle for over two thousand years, from prior to 500 B.C., all the way up until 1882. In 1882 it was shown that the problem of squaring the circle *cannot be solved.*

But what does this have to do with π? It turns out that the squaring the circle problem can be solved if and only if it is possible, using only straight edge and compass, to draw a line segment of length π. But it is not difficult to show that we can draw a line segment of length L, only if L is an algebraic number. An *algebraic* number is a number x such that there exists integers $a_0, \ldots, a_n$, with $a_n \neq 0$, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

That is, an algebraic number is one that is the root of a non-trivial polynomial with integer coefficients. If a number is not algebraic, then we say that it is transcendental. So to put these ideas together we have the following: *If π is a transcendental number, then the circle cannot be squared.*

In 1873, Charles Hermite proved that the number e is transcendental. In 1882 F. Lindemann extended this result to show that if r and s are distinct *complex* algebraic numbers, then the expression $e^r + e^s$ cannot be equal to zero. But Euler had already proven his famous equation: $e^{i\pi} + 1 = 0$. Letting $r = i\pi$ and $s = 0$, we can write $e^r + e^s = 0$. Since $s = 0$ is an algebraic number, the only possibility is that $r = i\pi$ is a transcendental number. But i is an algebraic number. We are forced to conclude that π is a transcendental number.

Proofs

So much for history. The rest of this article is devoted to giving some simple, elementary proofs of Ideas 1 through 5 above.

Ideas 1 and 2. I claim that Ideas 1 and 2 are actually quite obvious, once you think about them for a moment. (I do *not* claim this about Idea 3 though.) I will give an informal proof of Ideas 1 and 2. My informal proof can be turned into a formal proof in a number of different ways. Let us fix some circle Ω. Let C be the circumference of Ω, A the area of Ω, and r the radius of Ω. Let $k = C/r$ and let $h = A/r^2$. Now let Ω' be any other circle, let A' be its area, C' its circumference, and r' its radius. We want to see that $C'/r' = k$ and $A'/r' = h$. But I claim that this is obvious, because Ω' can be obtained by simply *expanding* or *contracting* Ω.

Let me be a bit more rigorous. Let us introduce a coordinate system to aid in the discussion. The coordinate system is not essential to my proof, but it will simplify the discussion. In order to exhibit how elementary and intuitive my proof is, I will make as little use of analytic geometry as possible. Since both area and arc length are intuitively preserved by "rigid motions," we may as well assume that both of our circles are centered at the origin. Also, to be concrete, let us assume that Ω' is a bigger circle than Ω, i.e. that $r' > r$. Let $\alpha = r'/r$. By the term "circle" we mean the set of all points a given distance from a center point. Thus Ω is the set of all points located a distance r from the origin, and Ω' is the set of all points located a distance r' from the origin. Consider the transformation T of the coordinate plane which sends a point (x, y) to the point $(\alpha x, \alpha y)$. T is an "expansion" by a factor of α. Suppose that (x, y) is a point on the circle Ω. Then the distance from (x, y) to the origin is r. T sends the point (x, y) to the point $(\alpha x, \alpha y)$. Using some simple facts about similar triangles, we see that the distance from $(\alpha x, \alpha y)$ to the origin is $\alpha r = r'$. Thus $(\alpha x, \alpha y)$ lies on the circle Ω'. Thus the transformation T maps Ω onto Ω'. Let us see what T does to area and arc length. Again using some simple facts about similar triangles, it is easy to see that if (x_1, y_1) and (x_2, y_2) are any two points in the plane and the distance between them is d, then the distance between $(\alpha x_1, \alpha y_1)$ and $(\alpha x_2, \alpha y_2)$ is αd. That is, T increases all distances by a factor of α. It follows that T increases the length of any polygonal path by a factor of α. Since our intuitive notion of arc length corresponds to the limit of the lengths of approximating polygonal paths, it follows that T increasing all arc lengths by a factor of α. As T sends Ω onto Ω' it follows that $C' = \alpha C$. Thus

$$\frac{C'}{r'} = \frac{\alpha C}{\alpha r} = \frac{C}{r} = k.$$

Also, since T increases all distances by a factor of α, T increases the area of any square by a factor of α^2. Since our intuitive notion of area corresponds to the limit of the areas of approximating collections of squares, T increases all areas by a factor of α^2. As T sends Ω onto Ω' it follows that $A' = \alpha^2 A$. Thus

$$\frac{A'}{(r')^2} = \frac{\alpha^2 A}{(\alpha r)^2} = \frac{A}{r^2} = h.$$

In summary, Ideas 1 and 2 follow immediately from the fact that expansions and contractions act linearly on the distance between two points. The fact that this is so is an integral part of our geometric intuition. It is the reason why two similar triangles have the same ratios of side lengths.

Idea 3. Unlike Ideas 1 and 2, I do not claim that Idea 3 is obvious. The proof of Idea 3 requires a more detailed analysis. Below I will give some arguments which will reprove Ideas 1 and 2, and also yield a proof of Idea 3.

My goal is to derive Ideas 1, 2, and 3 using "geometrically intuitive" reasoning. At several points I will even appeal to a diagram to make my argument. It is well known that such appeals to diagrams are dangerous, as diagrams can often be misleading. To give a completely rigorous geometric proof, I would have to give a list of geometric *axioms*, and derive all of my results from these axioms. I will not do this here, because I would like to keep this article short and simple. The interested reader is invited to try to translate my proof into a completely rigorous proof based on a set of axioms.

Definition 1. Let $A(r) =$ the area of the circle of radius r.

To understand my proof, pretend again that you have never heard of the number π. Now I will *define* π for you.

Definition 2. Let $\pi = A(1)$.

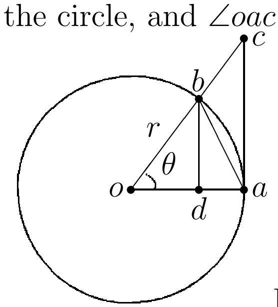
In the proof below I will mention the trigonometric functions sin, cos and tan. Since I intend to use only "geometrically intuitive" reasoning, I would like to point out that these trigonometric functions can be defined on acute angles, with no other assumptions besides the fact that similar triangles have the same ratio of side-lengths.

For convenience, I will use radians to measure angles. In so doing, I must be careful to avoid "begging the question." When we first encounter radians in a mathematics class there are two important facts we learn: (i) There are 2π radians of angle measure in a full circle, and (ii) An angle of one radian inscribed in a circle of radius r cuts off a circular arc of length r. From these two properties of radian measure we could conclude immediately that the circumference of a circle is given by $C = 2\pi r$. This of course would be a "circular argument." That is to say, it would be cheating. The reason is the following: Fact (ii) is usually taken as the *definition* of radian measure. Then, the formula $C = 2\pi r$ is used to see that fact (i) is true. So it would be cheating to use facts (i) and (ii) in order to prove that $C = 2\pi r$. To avoid this problem I will *not assume* fact (ii) above. Instead I will *define* radian measure so that fact (i) above holds. That is, let us define one *radian* to be the same angle measure as $[360 \div (2\pi)]^\circ$. Later, when we have proven the formulas $A = \pi r^2$ and $C = 2\pi r$, then fact (ii) above will follow.

Lemma 3. *For any real number $r > 0$ and any real number θ with $0 < \theta < \pi/2$, we have the following:*

$$\pi r^2 \frac{\sin\theta}{\theta} \leq A(r) \leq \pi r^2 \frac{\tan\theta}{\theta}.$$

Proof. Consider the following diagram, in which o is the center of the circle, and $\angle oac$ is a right angle.



Notice that the area of the triangle oab is less than the area of the circular segment oab which is less than the area of the triangle oac. Since $\overline{bd} = r\sin\theta$, the area of triangle oab is $\frac{1}{2}r^2\sin\theta$. The area of the circular segment oab is $\frac{\theta}{2\pi}A(r)$. The area of triangle oac is $\frac{1}{2}\tan\theta$. Thus we have:

$$\frac{1}{2}r^2\sin\theta \leq \frac{\theta}{2\pi}A(r) \leq \frac{1}{2}r^2\tan\theta.$$

Multiplying through by $\frac{2\pi}{\theta}$ we have

$$\pi r^2 \frac{\sin\theta}{\theta} \leq A(r) \leq \pi r^2 \frac{\tan\theta}{\theta}$$

which is what we were trying to prove. □

From the above lemma we can immediately derive a famous limit theorem.

Corollary 4. *If θ is measured in radians, then*

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Proof. Recall that we have defined $\pi = A(1)$. Setting $r = 1$ in the previous lemma we get:

$$\pi\frac{\sin\theta}{\theta} \leq \pi \leq \pi\frac{\tan\theta}{\theta}.$$

Canceling the term π we get

$$\frac{\sin\theta}{\theta} \leq 1 \leq \frac{\sin\theta}{\theta}\frac{1}{\cos\theta}.$$

Multiplying through by $\frac{\theta}{\sin\theta}$ yields

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}.$$

Inverting the fractions and reversing the inequalities yields:

(1) $$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

Just by thinking about right triangles, it is easy to see that as θ gets closer and closer to 0, $\cos\theta$ gets closer and closer to the value 1. In symbols we write this as

(2) $$\lim_{\theta\to 0^+} \cos\theta = 1.$$

From formulas 1 and 2 it follows that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

which is what we were trying to prove. □

From the previous two results we can derive the formula for the area of a circle.

Theorem 5. *For all $r > 0$, $A(r) = \pi r^2$.*

Proof. By taking the limit as θ goes to 0 in Lemma 3 and then applying Corollary 4 we get:

$$\pi r^2 \leq A(r) \leq \pi r^2.$$

□

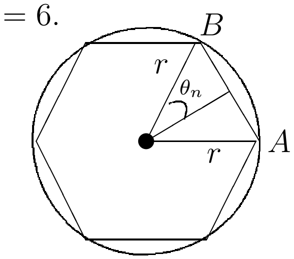
Now we turn to the circumference of a circle.

Definition 6. Let $C(r)$ = the circumference of the circle of radius r.

The next theorem will complete the proof of Ideas 1, 2, and 3.

Theorem 7. *For all $r > 0$, $C(r) = 2\pi r$.*

Proof. The circumference of a circle is equal to the limit as n goes to infinity of the perimeter of an inscribed regular n-gon. Let us calculate this perimeter. The following figure illustrates the situation in the case $n = 6$.



Let $\theta_n = \pi/n$. In the figure, we have $\sin\theta_n = (\frac{1}{2}\overline{AB})/r$. So $\frac{1}{2}\overline{AB} = r\sin\theta_n$. So $\overline{AB} = 2r\sin\theta_n$. So the perimeter of an inscribed regular n-gon is $n2r\sin\theta_n = 2\pi r(\sin\theta_n)/\theta_n$. So

$$C(r) = \lim_{n\to\infty} 2\pi r\frac{\sin\theta_n}{\theta_n} = \lim_{\theta\to 0} 2\pi r\frac{\sin\theta}{\theta} = 2\pi r.$$

□

Ideas 4 and 5. Finally, we will discuss one elementary way to calculate approximations to the number π. In Lemma 3 above, let $r = 1$ and let $\theta = \pi/n$. The result is the following inequality:

$$(3) \qquad n\sin\left(\frac{\pi}{n}\right) \leq \pi \leq n\tan\left(\frac{\pi}{n}\right).$$

If we let $n = 6$ we get

$$6 \times \frac{1}{2} \leq \pi \leq 6 \times \frac{1}{\sqrt{3}}.$$

If we then use the approximation $1.73 \leq \sqrt{3}$, we arrive at the rough approximation

$$3 \leq \pi \leq 3.5.$$

This approximation is rough, but at least we have *proved* it, and using very elementary techniques.

The inequalities in 3 can in principle be used to estimate π with as much accuracy as we desire. If we make the integer n larger and larger, we will get more and more accurate estimates. Of course this still leaves us with the problem of estimating $\sin(\pi/n)$ and $\tan(\pi/n)$. Notice that the solution "use a calculator to estimate $\sin(\pi/n)$ and $\tan(\pi/n)$" is of no help at all. We could of course use our calculator to immediately find an estimation of π to 8 decimal places. But then we wouldn't understand how the calculator is arriving at this estimate. The whole point is to prove our estimate of π using elementary techniques. The *half-angle* formulas from trigonometry will come to our rescue:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \quad \text{and} \quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

Since we know $\sin(\pi/6)$ and $\cos(\pi/6)$, using the half-angle formulas repeatedly will allow us to calculate $\sin(\pi/n)$, $\cos(\pi/n)$, and $\tan(\pi/n)$, for $n = 6, 12, 24, 48, 96, \cdots$. With $n = 12$ we get the estimate

$$12 \times \frac{\sqrt{2-\sqrt{3}}}{2} \leq \pi \leq 12 \times (2-\sqrt{3})$$

and by approximating the square roots we get $3.1 \leq \pi \leq 3.22$. With $n = 24$ we get the estimate

$$24 \times \frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2} \leq \pi \leq 24 \times \left(2\sqrt{2+\sqrt{3}} - 2 - \sqrt{3}\right)$$

and by approximating the square roots we get $3.13 \leq \pi \leq 3.16$. As you can see, our method of approximating π is not very *efficient*. There are far more efficient algorithms for estimating π, but we will not go into them here.

A Different Point of View

In the previous section I took the point of view that it is desirable to prove the formulas $A = \pi r^2$ and $C = 2\pi r$ using geometrically intuitive reasoning, and using as little abstract analysis as possible. It is also possible to take the opposite point of view, namely that it is desirable to prove the two formulas using *no* geometric reasoning, and using only abstract analysis. I conclude this article with a quick sketch of how one might do this.

To begin with, one can *define* the circle of radius r to be the graph of the equation $x^2 + y^2 = r^2$. Then, letting $f(x) = \sqrt{r^2 - x^2}$, one can *define* the area and circumference of this circle with the formulas

$$A(r) = 4\int_0^r f(x)dx \quad \text{and} \quad C(r) = 4\int_0^r \sqrt{1 + \big(f'(x)\big)^2}dx.$$

Then one can *define* the sine and cosine functions by the familiar power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

With these definitions alone, and without appealing to geometry at all, one can prove all of the familiar properties of the trigonometric functions. In particular one can prove that for all x, $\sin^2 x + \cos^2 x = 1$, that sin and cos are *periodic* functions, and that the derivative of the function $\sin x$ is $\cos x$, and the derivative of the function $\cos x$ is $-\sin x$. Then one can *define* the number π as half the period of the sin function. Then one can prove that $\sin(\frac{\pi}{2}) = 1$, that $\sin(-\frac{\pi}{2}) = -1$ and that $\sin x$ is a one-to-one function on the interval $[-\pi/2, \pi/2]$. Now let $\sin^{-1} x$ be the inverse of the function $\sin x$ over the interval $[-\pi/2, \pi/2]$. So $\sin^{-1}(1) = \pi/2$, and $\sin^{-1}(0) = 0$. Letting $F(x) = \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + \frac{x}{2}\sqrt{r^2 - x^2}$ and letting $G(x) = r\sin^{-1}(x/r)$ one can prove that $F'(x) = \sqrt{r^2 - x^2} = f(x)$ and $G'(x) = r\big(r^2 - x^2\big)^{-1/2} = \sqrt{1 + \big(f'(x)\big)^2}$. Finally, one can conclude that

$$A(r) = 4\int_0^r f(x)dx = 4\big[F(x)\big]_0^r = 4\big[\tfrac{r^2}{2}\sin^{-1}(1)\big] = 4\Big[\frac{r^2}{2}\frac{\pi}{2}\Big] = \pi r^2$$

and that

$$C(r) = 4\int_0^r \sqrt{1 + \big(f'(x)\big)^2}dx = 4\big[G(x)\big]_0^r = 4r\big[\sin^{-1}(1)\big] = 4r\Big[\frac{\pi}{2}\Big] = 2\pi r.$$

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[1] Petr Beckmann. *A History of Pi*, fourth edition. The Golem Press, Boulder, CO, 1977.

[2] David M. Burton. *Burton's History of Mathematics: An Introduction*, third edition. Wm. C. Brown Publishers, Dubuque, IA, 1995.