Theory of Algorithms. Spring 2000. Homework 9 Solutions.

Section 8.1

(5a) Let $L = \left\{ a^n w w^R a^n \,:\, n \geq 0, w \in \{a,b\}^* \right\}$. Then L **is** context-free. In fact, L is just $\left\{ w w^R \,:\, w \in \{a,b\}^* \right\}$.

(5b) Let $L = \{ a^n b^j a^n b^j \,:\, n \geq 0, j \geq 0 \}$. Then L is not context-free.

Proof. Assume towards a contradiction that L is context-free. Let $m > 0$ be given by the Context-Free Pumping Lemma. Then let $w = a^m b^m a^m b^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = uvxyz$ be the decomposition of w given by the Pumping Lemma. So $|vxy| \leq m$ and $|vy| \geq 1$. Here are some of the possibilities for what v and y look like:

(Case 1) vxy is a substring of the first block of a's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+k}b^m a^m b^m$ where $k = |vy|$. Since $k \neq 0$, $w_2 \notin L$.

(Case 2) vx is a substring of the first block of a's, and $y = a^k b^s$ for some $k, s \geq 1$. In this case let $i = 2$. Notice that $w_2 \notin L(a^*b^*a^*b^*)$, so $w_2 \notin L$.

(Case 3) v is a substring of the first block of a's and y is a substring of the first block of b's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+k}b^{m+s}a^m b^m$ where $k = |v|$ and $s = |y|$. Since k and s cannot both be 0, we have that $w_2 \notin L$.

All other cases for what v and y look like are similar to one of the above cases. So in all cases we can find an i so that $w_i \notin L$. This contradicts the Pumping Lemma. So L is not context-free. □

(5c) Let $L = \{ a^n b^j a^j b^n \,:\, n \geq 0, j \geq 0 \}$. Then L **is** context-free. Here is a context-free grammar for L:

$$\begin{aligned} S &\to aSb \,|\, A \\ A &\to bAa \,|\, \lambda \end{aligned}$$

(5d) Let $L = \left\{ a^n b^j a^k b^l \,:\, n + j \leq k + l \right\}$. Then L **is** context-free. Here is a context-free grammar for L:

$$\begin{aligned} S &\to aSB \,|\, S_1 \,|\, S_2 \\ S_1 &\to aS_1A \,|\, S_3 \,|\, \lambda \\ S_2 &\to bS_2B \,|\, S_3 \,|\, \lambda \\ S_3 &\to bS_3A \,|\, \lambda \\ A &\to AA \,|\, a \\ B &\to BB \,|\, b \end{aligned}$$

(5e) Let $L = \left\{ a^n b^j a^k b^l \ : \ n \leq k \ , \ j \leq l \right\}$. Then L is not context-free.

Proof. Assume towards a contradiction that L is context-free. Let $m > 0$ be given by the Context-Free Pumping Lemma. Then let $w = a^m b^m a^m b^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = uvxyz$ be the decomposition of w given by the Pumping Lemma. So $|vxy| \leq m$ and $|vy| \geq 1$. Here are some of the possibilities for what v and y look like:

(Case 1) vxy is a substring of the first block of a's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+s}b^m a^m b^m$ where $s = |vy|$. Since $s > 0$, $w_2 \notin L$.

(Case 2) vxy is a substring of the *second* block of a's. In this case let $i = 0$. We have $w_0 = uxz = a^m b^m a^{m-s} b^m$ where $s = |vy|$. Since $s > 0$, $w_0 \notin L$.

(Case 3) vx is a substring of the first block of a's, and $y = a^s b^t$ for some $s, t \geq 1$. In this case let $i = 2$. Notice that $w_2 \notin L(a^*b^*a^*b^*)$, so $w_2 \notin L$.

(Case 4) v is a substring of the first block of a's and y is a substring of the first block of b's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+s}b^{m+t}a^m b^m$ where $s = |v|$ and $t = |y|$. Since s and t cannot both be 0, we have that $w_0 \notin L$.

(Case 5) v is a substring of the *second* block of a's and y is a substring of the *second* block of b's. In this case let $i = 0$. We have $w_2 = uxz = a^m b^m a^{m-s} b^{m-t}$ where $s = |v|$ and $t = |y|$. Since s and t cannot both be 0, we have that $w_0 \notin L$.

All other cases for what v and y look like are similar to one of the above cases. So in all cases we can find an i so that $w_i \notin L$. This contradicts the Pumping Lemma. So L is not context-free. □

(5f) Let $L = \{ a^n b^n c^j \ : \ n \leq j \}$. Then L is not context-free.

Proof. Assume towards a contradiction that L is context-free. Let $m > 0$ be given by the Context-Free Pumping Lemma. Then let $w = a^m b^m c^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = uvxyz$ be the decomposition of w given by the Pumping Lemma. So $|vxy| \leq m$ and $|vy| \geq 1$. Here are some of the possibilities for what v and y look like:

(Case 1) vxy is a substring of the block of a's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+s}b^m c^m$ where $s = |vy|$. Since $s > 0$, $w_2 \notin L$.

(Case 2) vxy is a substring of the block of c's. In this case let $i = 0$. We have $w_0 = uxz = a^m b^m c^{m-s}$ where $s = |vy|$. Since $s > 0$, $w_0 \notin L$.

(Case 3) vx is a substring of the block of a's, and $y = a^s b^t$ for some $s, t \geq 1$. In this case let $i = 2$. Notice that $w_2 \notin L(a^*b^*a^*b^*)$, so $w_2 \notin L$.

(Case 4) v is a substring of the block of a's and y is a substring of the block of b's. In this case let $i = 2$. We have $w_2 = uv^2xy^2z = a^{m+s}b^{m+t}c^m$ where $s = |v|$ and $t = |y|$. Since s and t cannot both be 0, we have that $w_2 \notin L$.

(Case 5) v is a substring of the block of b's and y is a substring of the block of c's. In this case let $i = 0$. We have $w_2 = uxz = a^m b^{m-s} c^{m-t}$ where $s = |v|$ and $t = |y|$. Since s and t cannot both be 0, we have that $w_0 \notin L$.

All other cases for what v and y look like are similar to one of the above cases. So in all cases we can find an i so that $w_i \notin L$. This contradicts the Pumping Lemma. So L is not context-free. $\square$