Theory of Algorithms. Spring 2000. Homework 7 Solutions.

Section 5.1

(4) Let $L = \{\, ab(bbaa)^n bba(ba)^n \; : \; n \geq 0 \,\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = ab(bbaa)^m bba(ba)^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that there are many different possibilities for what y looks like and therefore it would seem as if this proof is going to be difficult. But in fact it will be easy. Let $i = 2$. I claim that $w_2 \notin L$. To see this, I argue as follows. Notice that $bba(ba)^m$ is a suffix of w and so it is also a sufix of w_2. But now notice that our string w is the one and only string in L that has $bba(ba)^m$ as a suffix. Since $w_2 \neq w$ and yet w_2 has $bba(ba)^m$ as a suffix, $w_2 \notin L$. This contradicts the Pumping Lemma. So L is not regular. □

(6) (a) $S \to aSb \mid aaa \mid aa \mid a \mid \lambda$

(b)
$$S \to aSb \mid A \mid B \mid \lambda$$
$$A \to aA \mid \lambda$$
$$B \to bB \mid bb$$

(c)
$$S \to S_1 \mid S_2$$
$$S_1 \to aaS_1b \mid aS_1 \mid a$$
$$S_2 \to AAs_2b \mid Ab$$
$$A \to a \mid \lambda$$

(d)
$$S \to aSbbB \mid \lambda$$
$$B \to b \mid \lambda$$

As preparation for part (e) let us review exercise 14(b) from section 1.2 (Homework 1.) In that exercise you were asked for a grammar for the language $L = \{\, w \in \{a, b\}^* \; : \; n_a(w) > n_b(w) \,\}$. My answer was the following grammar G.

$$S \to aS_1 \mid aS \mid S_1S$$
$$S_1 \to S_1S_1 \mid \lambda \mid aS_1b \mid bS_1a$$

Let me explain why it is true that $L(G) = L$. First notice that by Example 1.12 on page 23, $S_1 \Rightarrow^* w$, iff $n_a(w) = n_b(w)$. It is then clear that $L(G) \subseteq L$, because the only way to eliminate the variable S is via the rule $S \to aS_1$. This adds one a and no b.

So we must see that $L \subseteq L(G)$. Now suppose $w \in L$. We must see that $S \Rightarrow^* w$. We argue by induction on $|w|$. As the basis of the induction, the shortest string in L is the string a, and it is true that $S \Rightarrow^* a$. Now suppose that $|w| \geq 2$.

If w starts with a b then by an argument like the argument given in Example 1.12 on page 23, w must have a non-empty prefix w_1 such that $n_a(w_1) = n_b(w_1)$. Now $S_1 \Rightarrow^* w_1$, and $w = w_1w_2$ with w_2 in L. Since $|w_2| < |w|$, by induction we know that $S \Rightarrow^* w_2$. Thus $S \Rightarrow S_1S \Rightarrow^* w_1w_2 = w$.

So now suppose $|w| \geq 2$ and $w = aw_1$ for some string w_1. Then $n_a(w_1) \geq n_b(w_1)$. If $n_a(w_1) = n_b(w_1)$ then $S_1 \Rightarrow^* w_1$ and so $S \Rightarrow aS_1 \Rightarrow^* aw_1 = w$. If $n_a(w_1) > n_b(w_1)$ then $w_1 \in L$, and since $|w_1| < |w|$ by induction we know that $S \Rightarrow^* w_1$. So $S \Rightarrow aS \Rightarrow^* aw_1 = w$.

Given the previous discussion, I hope you will understand the solution to part (e) below.

(e)
$$\begin{aligned} S &\to S_a \,|\, S_b \\ S_a &\to aS_1 \,|\, aS_a \,|\, S_1S_a \\ S_b &\to bS_1 \,|\, bS_b \,|\, S_1S_b \\ S_1 &\to S_1S_1 \,|\, \lambda \,|\, aS_1b \,|\, bS_1a \end{aligned}$$

(f)
$$S \to aS \,|\, aSbS \,|\, \lambda$$

Now I explain my solution to part (f). Let G be the grammar in part (f) above and let L the language given in the textbook exercise. I will show that $L(G) = L$. By induction on the length of a derivation of a sentential form u, we can see that every sentential form u has the property that if v is a prefix of u then $n_a(v) \geq n_a(u)$. This proves that $L(G) \subseteq L$. So now we must see that $L \subseteq L(g)$. We will prove by induction on the length of w that if $w \in L$ then $S \Rightarrow^* w$. As the basis step of the induction, if $w = \lambda$ then indeed $S \Rightarrow^* w$. So now suppose $|w| \geq 1$. Notice that it must be true that the first symbol of w is a. So let us write $w = aw_1$. If $w_1 \in L$ then since $|w_1| < |w|$ we have by induction that $S \Rightarrow^* w_1$ and so $S \Rightarrow aS \Rightarrow^* aw_1 = w$. So now suppose that $w_1 \notin L$. Then there is some prefix v of w_1 such that $n_b(v) > n_a(v)$. Let v be the shortest such prefix. Notice then that the final symbol of v is b and that and $n_b(v) = n_a(v) + 1$. Let us write $v = xb$. Notice that $x \in L$ and $n_a(x) = n_b(x)$. Also let us write $w_1 = vy = xby$. Notice then that $w = aw_1 = axby$. Now $n_a(axb) = n_b(axb)$ and so we must have that $y \in L$. Since x and y are in L and $|x|, |y| < |w|$, we have by induction that $S \Rightarrow^* x$ and $S \Rightarrow^* y$. It then follows that $S \Rightarrow aSbS \Rightarrow^* axby = w$.

(7a)
$$\begin{aligned} S &\to S_1C \,|\, AS_2 \\ S_1 &\to aS_1b \,|\, \lambda \\ S_2 &\to bS_2c \,|\, C \,|\, \lambda \\ A &\to aA \,|\, \lambda \\ C &\to cC \,|\, \lambda \end{aligned}$$

(b)
$$\begin{aligned} S &\to S_1C \,|\, AS_2 \\ S_1 &\to aS_1b \,|\, \lambda \\ S_2 &\to bS_2c \,|\, bB \,|\, cC \,|\, \lambda \\ A &\to aA \,|\, \lambda \\ B &\to bB \,|\, \lambda \\ C &\to cC \,|\, \lambda \end{aligned}$$

(c)
$$\begin{aligned} S &\to aSc|A \\ A &\to bAc \,|\, \lambda \end{aligned}$$

Section 7.1

(4a) $z = 0,\ F = \{q_3\}$

$$\delta(q_0, \lambda, 0) = \{(q_3, \lambda)\}$$
$$\delta(q_0, a, 0) = \{(q_1, 110)\}$$
$$\delta(q_1, a, 1) = \{(q_1, 111)\}$$
$$\delta(q_1, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, \lambda, 0) = \{(q_3, \lambda)\}$$

(4b) $F = \{q_2\}$

$$\delta(q_0, a, z) = \{(q_0, az)\}$$
$$\delta(q_0, b, z) = \{(q_0, bz)\}$$
$$\delta(q_0, a, a) = \{(q_0, aa)\}$$
$$\delta(q_0, b, a) = \{(q_0, ba)\}$$
$$\delta(q_0, a, b) = \{(q_0, ab)\}$$
$$\delta(q_0, b, b) = \{(q_0, bb)\}$$

$$\delta(q_0, c, z) = \{(q_1, z)\}$$
$$\delta(q_0, c, a) = \{(q_1, a)\}$$
$$\delta(q_0, c, c) = \{(q_1, b)\}$$

$$\delta(q_1, a, a) = \{(q_1, \lambda)\}$$
$$\delta(q_1, b, b) = \{(q_1, \lambda)\}$$
$$\delta(q_1, \lambda, z) = \{(q_2, z)\}$$

(4c) $z = 0,\ F = \{q_3\}$

$$\delta(q_0, \lambda, 0) = \{(q_3, \lambda)\}$$
$$\delta(q_0, a, 0) = \{(q_0, 10)\}$$
$$\delta(q_0, a, 1) = \{(q_0, 11)\}$$
$$\delta(q_0, b, 0) = \{(q_1, 10)\}$$
$$\delta(q_0, b, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, b, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, c, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, c, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, \lambda, 0) = \{(q_3, \lambda)\}$$

(4g) $F = \{q_2\}$

$$\delta(q_0, b, z) = \{(q_0, 0z)\}$$
$$\delta(q_0, b, 0) = \{(q_0, 00)\}$$

$$\delta(q_0, a, z) = \{(q_1, z)\}$$
$$\delta(q_0, z, 0) = \{(q_1, 0)\}$$

$$\delta(q_1, b, 0) = \{(q_1, 00)\}$$
$$\delta(q_1, b, z) = \{(q_1, 0z)\}$$
$$\delta(q_1, b, 1) = \{(q_1, \lambda)\}$$

$$\delta(q_1, a, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, a, z) = \{(q_1, 1z)\}$$
$$\delta(q_1, a, 0) = \{(q_1, \lambda)\}$$

$$\delta(q_1, \lambda, z) = \{(q_2, z)\}$$

(4h) $F = \{q_2\}$

$$\delta(q_0, a, z) = \{(q_0, 1z)\}$$
$$\delta(q_0, a, 1) = \{(q_0, 11)\}$$
$$\delta(q_0, a, 0) = \{(q_0, \lambda)\}$$

$$\delta(q_0, b, z) = \{(q_0, 00z)\}$$
$$\delta(q_0, b, 0) = \{(q_0, 000)\}$$
$$\delta(q_0, b, 1) = \{(q_1, \lambda)\}$$

$$\delta(q_1, \lambda, 1) = \{(q_0, \lambda)\}$$
$$\delta(q_1, \lambda, 0) = \{(q_0, 00)\}$$
$$\delta(q_1, \lambda, z) = \{(q_0, 0z)\}$$

$$\delta(q_0, \lambda, z) = \{(q_2, z)\}$$

(4i) $F = \{q_1\}$

$$\delta(q_0, a, z) = \{(q_0, 1z)\}$$
$$\delta(q_0, a, 1) = \{(q_0, 11)\}$$
$$\delta(q_0, a, 0) = \{(q_0, \lambda)\}$$
$$\delta(q_0, b, z) = \{(q_0, 1z)\}$$
$$\delta(q_0, b, 1) = \{(q_0, 11)\}$$
$$\delta(q_0, b, 0) = \{(q_0, \lambda)\}$$

$$\delta(q_0, c, z) = \{(q_0, 0z)\}$$
$$\delta(q_0, c, 0) = \{(q_0, 00)\}$$
$$\delta(q_0, c, 1) = \{(q_0, \lambda)\}$$

$$\delta(q_0, \lambda, z) = \{(q_1, z)\}$$

(5) $F = \{q_f\}$

$$\delta(q_0, \lambda, z) = \{(q_1, z), (q_4, z)\}$$

$$\delta(q_1, a, z) = \{(q_2, z)\}$$

$$\delta(q_2, a, z) = \{(q_2, z), (q_2, 1z)\}$$
$$\delta(q_2, b, 1) = \{(q_3, \lambda)\}$$
$$\delta(q_3, b, 1) = \{(q_3, \lambda)\}$$
$$\delta(q_3, \lambda, z) = \{(q_f, z)\}$$

$$\delta(q_4, a, z) = \{(q_4, 1z)\}$$
$$\delta(q_4, a, 1) = \{(q_4, 11)\}$$
$$\delta(q_4, b, z) = \{(q_5, z)\}$$
$$\delta(q_4, b, 1) = \{(q_5, 1)\}$$
$$\delta(q_5, b, 1) = \{(q_5, 1), (q_5, \lambda)\}$$
$$\delta(q_5, \lambda, z) = \{(q_f, z)\}$$

(10) Note that there is never more than one symbol on the stack. So this npda is equivalent to an nfa. The language accepted is given by the regular expression $a + abb^*a$.

(11) Every string is accepted. The language is $\{a, b\}^*$.

(12) $L(\lambda + a + abb^* + abb^*a)$.

(14) $\{a\}$.