Theory of Algorithms. Spring 2000. Homework 1 Solutions.

Section 1.2

(4) All but the third string are in L^*.

(5) If $L = \{ a^n b^{n+1} : n \geq 0 \}$ then $L \neq L^*$. For example, $\lambda \in L^*$ but $\lambda \notin L$.
If $L = \{ w : n_a(w) = n_b(w) \}$ then $L = L^*$.

(6) There are no languages L such that $\overline{L^*} = \bar{L}^*$. One way to see this is that for any language L, $\lambda \in \bar{L}^*$ but $\lambda \notin \overline{L^*}$.

(8) (a)
$$S \to BaB$$
$$B \to bB \mid \lambda$$

(b)
$$S \to AaA$$
$$A \to aA \mid bA \mid \lambda$$

(c)
$$S \to B \mid BaB \mid BaBaB \mid BaBaBaB$$
$$B \to bB \mid \lambda$$

(9) $L(G) = \{ (ab)^n \mid n \geq 0 \}$.

(10) $L(G) = \emptyset$ because there is no way to derive a string that doesn't contain one of the auxiliary symbols S, A, B.

(11) (a)
$$S \to aSb \mid B$$
$$B \to bB \mid b$$

(b) $S \to aSbb \mid \lambda$.

(c)
$$S \to aaaAb$$
$$A \to aAb \mid \lambda$$

(d)
$$S \to aaaA$$
$$A \to aAb \mid \lambda$$

(e)
$$S \to S_1 S_2$$
$$S_1 \to aS_1 b \mid B$$
$$B \to bB \mid b$$
$$S_2 \to aS_2 bb \mid \lambda$$

(f)
$$S \to S_1 \mid S_2$$
$$S_1 \to aS_1 b \mid B$$
$$B \to bB \mid b$$
$$S_2 \to aS_2 bb \mid \lambda$$

(g)
$$S \to S_1 S_1 S_1$$
$$S_1 \to aS_1 b \mid B$$
$$B \to bB \mid b$$

(h)
$$S \to SS \mid \lambda \mid S_1$$
$$S_1 \to aS_1 b \mid B$$
$$B \to bB \mid b$$

(i) $L_1 - \bar{L}_4 = L_1 \cap L_4$. But $L_1 \cap L_4 = \emptyset$. A grammar for the empty language is: $S \to S$.

(12) (a) $S \to aaaS \mid \lambda$.

(c) $S \to aaaaaaS \mid aa \mid aaa \mid aaaa \mid aaaaa$

(13) $S \to aSa \mid bSb \mid aa \mid bb$

CLAIM Let G be the grammar above. Let $L = \left\{ ww^R \,\middle|\, w \in \{a,b\}^+ \right\}$. The $L(G) = L$.

Proof. First we show that $L(G) \subseteq L$. We prove by induction on the length of a derivation of v that if v is a sentential form of G then v has one of the following three forms: (i) S; (ii) wSw^R; or (iii) ww^R; where $w \in \{a,b\}^+$.
Basis Step. If the derivation of v has length 0 then $v = S$.
Inductive Step. Suppose $S \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_n \Rightarrow v$ is a derivation of v of length $n+1$. By induction, v_n has form (i) (ii) or (iii). Since all production rules have an S on the left hand side, v_n cannot have form (iii). If $v_n = S$ then by examining the production rules we see that $v = aSa$ or $v = bSb$ (these are of form (ii)) or $v = aa$ or $v = bb$ (these are of form (iii)). If $v_n = wSw^R$ then by examining the production rules we see that $v = waSaw^R$ or $v = wbSbw^R$ or $v = waaw^R$ or $v = wbbw^R$. Since $aw^R = (wa)^R$ and $bw^R = (wb)^R$, we are done.

Next we will prove that $L \subseteq L(G)$. We prove by induction on the length of a string $w \in \{a,b\}^+$ that wSw^R is a sentential form of G. Since G contains the production rule $s \to \lambda$, this is sufficient.
Basis Step. If $|w| = 1$ then $w = a$ or $w = b$. By inspection, aSa and bSb are sentential forms of G.
Inductive Step. Suppose $|w| = n+1$ where $n \geq 1$. Then there is a string v of length n such that either $w = va$ or $w = vb$. By induction vSv^R is a sentential form of G. By inspection then $vaSav^R$ and $vbSbv^R$ are sentential forms. Since $av^R = (va)^R$ and $bv^R = (vb)^R$, we are done. ☐

(14) (a)
$$S \to S_1 a S_1$$
$$S_1 \to S_1 S_1 \mid \lambda \mid aS_1 b \mid bS_1 a$$

(b)
$$S \to aS_1 \mid aS \mid S_1 S$$
$$S_1 \to S_1 S_1 \mid \lambda \mid aS_1 b \mid bS_1 a$$

(17) No, the 2 grammars are *not* equivalent. The first generates λ and the second does not.

(19) $L(G) = \left\{ a^{2^n} \,:\, n \geq 1 \right\}$.