PHY-6936: ADVANCED TOPICS (Advanced Quantum Theory III) (Dr R Fiebig)

The plan is to cover the topics listed below. The list is tentative and subject to change as the course proceeds. In broad terms the topics are discussed in Chapters 8-12 of *Ernest S. Abers, Quantum Mechanics*, see the bibliography below. Other sources from the bibliography may be consulted.

1 SCATTERING THEORY
1.1 Møller operators and S-matrix
1.2 Scattering states and propagators
1.3 T-operator and Lippmann-Schwinger equation
1.4 Partial wave expansion in potential scattering
1.5 Scattering amplitude and phase shifts
1.6 Regular solution and Volterra equation

2 TIME DEPENDENT PERTURBATION THEORY
2.1 Transitions in external field
2.2 Transition probability
2.3 Fermi's golden rule
2.4 Optical theorem
2.5 Decay of excited states

3 PATH INTEGRATION
3.1 Propagator in one dimension
3.2 Free particle propagator
3.3 Euclidean formalism

4 RELATIVISTIC WAVE EQUATION
4.1 Lorentz transformations
4.2 Representations
4.3 Klein-Gordon equation
4.4 Dirac equation
4.5 Dirac electron in electromagnetic field

5 CANONICAL FIELD QUANTIZATION
5.1 Quantization of a linear chain
5.2 Functional derivatives
5.3 Quantization of the Klein-Gordon field
5.4 Invariant propagator functions

Bibliography

- Ernest S. Abers, Quantum Mechanics, 2004 by Pearson Education Inc., ISBN 0-13-146100-1
- John R. Taylor, Scattering Theory,: The Quantum Theory on Nonrelativistic Collisions, John Wiley & Sons, Inc. 1972, ISBN 0-471-84900-6
- Roger G. Newton, Scattering Theory of Waves and Particles, McGraw Hill 1966
- Albert Messiah, Quantum Mechanics, Volume II, North-Holland, Amsterdam 1965
- Walter Greiner and Joachim Reinhardt, Field Quantization, Springer, New York, 1996, ISBN 3-540-59179-6
- Michael E. Peskin and Daniel V. Schroeder, An Introduction to Quantum Field Theory, Westview Press 1995, ISBN 0-201-50397-2

HOMEWORK

Problem 1

Consider the asymptotic form of a wave function in coordinate representation subject to scattering boundary conditions

$$\psi(\vec{x}) \longrightarrow (2\pi)^{-3/2}[e^{i\vec{p}\cdot\vec{x}} + f(\vec{p}\,' \leftarrow \vec{p})\frac{e^{ipr}}{r}] \quad \text{as} \quad r \longrightarrow \infty$$

where $r = |\vec{x}|$ and f is a complex function of the momenta $\vec{p}, \vec{p}\,'$ with $p = p'$.
(a) Separately calculate the probability current densities $\vec{j}_0$ and $\vec{j}_1$ of the plane and the spherical wave components, respectively.
(b) What is the number of counts n_0 per time in a detector of area $d\sigma$ placed in the incident beam?
(c) What is the number of counts n_1 per time in a detector of area da counting the scattered particles at distance r from the target region?
(d) The ratio n_1/n_0 is a measurable quantity. Calculate it in terms of f.

Problem 2

Work out problem 8.4 of [Abers].

Problem 3

An alternative form of the partial wave Lippmann-Schwinger equation for potential scattering is

$$\phi_{\ell p}(r) = \hat{j}_\ell(pr) + \int_0^r dr'\, g_{0,\ell p}(r, r')\, 2mV(r')\, \phi_{\ell p}(r')$$

where

$$g_{0,\ell p}(r, r') = \frac{1}{p}\left[\hat{j}_\ell(pr)\hat{n}_\ell(pr') - \hat{n}_\ell(pr)\hat{j}_\ell(pr')\right]\Theta(r - r')$$

and $\hat{j}_\ell(x) = xj_\ell(x), \hat{n}_\ell(x) = xn_\ell(x)$ and $\hat{h}_\ell^{(\pm)}(x) = \hat{n}_\ell(x) \pm i\hat{j}_\ell(x)$ are spherical Bessel, Neumann, and Hankel functions. The radial wave function $\phi_{\ell p}(r)$ is known as the regular solution.
(a) Prove that $g_{0,\ell p}(r, r')$ is a Green's function of the radial Schrödinger equation.
(b) Show that the asymptotic behavior of the regular solution is

$$\phi_{\ell p}(r) \rightarrow \frac{i}{2}\mathcal{F}_\ell(p)\hat{h}_\ell^{(-)}(pr) - \frac{i}{2}\mathcal{F}_\ell^*(p)\hat{h}_\ell^{(+)}(pr) \quad \text{as} \quad r \rightarrow \infty$$

where

$$\mathcal{F}_\ell(p) = 1 + \frac{1}{p}\int_0^\infty dr\, \hat{h}_\ell^{(+)}(pr)\, 2mV(r)\, \phi_{\ell p}(r)$$

is called the Jost function.
(c) Show that the partial wave S-matrix is

$$s_\ell(p) = e^{2i\delta_\ell(p)} = \frac{\mathcal{F}_\ell^*(p)}{\mathcal{F}_\ell(p)}\,.$$

You will need the asymptotic form of the Hankel functions.

Problem 4

A non-local potential is an operator V defined through

$$\langle \vec{x}|V|\psi\rangle = \int d^3\vec{x}\,'\langle \vec{x}|V|\vec{x}\,'\rangle\langle \vec{x}\,'|\psi\rangle\,.$$

Solve the scattering problem for a separable non-local potential V with the the integral kernel

$$\langle \vec{x}|V|\vec{x}\,'\rangle = \lambda\, g(\vec{x})g^*(\vec{x}\,')\,,$$

where g is a function with $||g|| = 1$. Pay attention to the sign of the coupling λ. Specifically calculate
(a) the off-shell and on-shell T-matrix elements
(b) the scattering amplitude
(c) the differential cross section.
(d) the scattering phase shifts for partial waves $\ell > 0$.

Problem 5

Work out problem 9.1 of [Abers]. For the external electromagnetic field choose a gauge with electric potential $\Phi = 0$.

Problem 6

Consider a nucleus at a fixed position (for example in a solid, or in human tissue). The magnetic moment operator then is

$$\vec{\mu} = \frac{Ze}{2M} g_s \vec{S}\,,$$

where M is the mass and $\vec{S}$ is the spin operator. The nucleus is placed in an external electromagnetic field given by

$$\vec{A} = \frac{1}{2}\int d\omega\, \Delta(\omega)\, [\vec{B}_0 + \vec{B}_1 \cos(\omega t)] \times \vec{r}\,.$$

The vectors $\vec{B}_0$ and $\vec{B}_1$ are both constant and have the same direction, and $\Delta(\omega)$ is a function strongly peaked around some frequency ω_0.
(a) What is the Hamiltonian of the system?
(b) Let $|jm>, m = -j \ldots + j$ be the eigenstates of S_z relative to the axis given by $\vec{B}_0$. Calculate the transition rates $\Gamma_{m_2 \leftarrow m_1}$ for pair of eigenstates of S_z.

Problem 7

A hydrogen atom in the ground state $|n\ell m >= |100>$ is subject to electron scattering. You may assume $m_e \ll m_p$. Take

$$V = -\frac{e^2}{r}\exp\left(-\epsilon r\right)$$

to be the Coulomb potential energy of the atom. The exp factor describes the "screening" of the proton's charge by the electron cloud. (For large distance the total charge of the atom is zero.) You should let $\epsilon \to 0$ after obtaining results.
(a) Apply Fermi's "Golden Rule" to calculate the probability for transitions into the excited state $|n\ell m>=|200>$.

Problem 8

Work out the derivation of the path integral representation

$$\langle q_b|e^{-iHT}|q_a\rangle = \int [dq][dp]\, \exp\left[i \int_0^T (p\dot{q} - H(p,q))\right]$$

for the Weyl-ordered Hamiltonian $H(p,q) = qp^2q$. You may assume one degree of freedom, coordinate $q = q^j, j = 1$ with conjugate momentum $p = p^j, j = 1$. Also take $T = t_b - t_a$. Use the standard time-slice discretization.

Problem 9

Calculate the Euclidean propagator $\langle q_b|e^{-HT}|q_a\rangle$ for a free particle $L(\dot{q}) = \frac{m}{2}\dot{q}^2$ using the path integral formalism directly in Euclidean time. Compare your result to the Minkowski (physical time) version.

Problem 10

The Euclidean propagator for the one-dimensional harmonic oscillator is

$$\langle q_b|e^{-HT}|q_a\rangle = \sqrt{\frac{m\omega}{2\pi\sinh(\omega T)}}\, \exp\left(-\frac{m\omega}{2\pi\sinh(\omega T)}[(q_a^2 + q_b^2)\cosh(\omega T) - 2q_a q_b]\right) .$$

where $T = t_b - t_a$. Use this to calculate the wave function of the first and second excited states, $n = 1, 2$.

Problem 11

Prove that the (Dirac) matrices

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$$

are solutions of

$$[\gamma^\nu, \sigma_{\alpha\beta}] = 2i(\gamma_\beta \delta^\nu{}_\alpha - \gamma_\alpha \delta^\nu{}_\beta) .$$

Problem 12

Consider a Dirac particle in an external electromagnetic field,

$$(i\partial\!\!\!/ - m)\psi = eA\!\!\!/\psi .$$

(a) Determine the properties of the operator and $\vec{\pi} = -i\vec{\nabla} + e\vec{A}$ under a local gauge

transformation $\vec{\pi} \to e^{-i\alpha(x)}\vec{\pi}e^{i\alpha(x)}$. How is $\vec{\pi}$ related to the covariant derivative?
(b) Let $\vec{x}$ be the position operator, then calculate the commutators $[\pi^i, x^j], i, j = 1, 2, 3$.
(c) Construct the 4×4 matrix operator H such that

$$i\frac{\partial\psi}{\partial t} = H\psi\,.$$

It is useful to write H interms of the matrices $\vec{\alpha}$ and β and the potentials $\vec{A}$ and A^0.
(d) Calculate the time derivatives

$$\frac{d\vec{x}}{dt} \quad \text{and} \quad \frac{d\vec{\pi}}{dt}$$

using Heisenberg's equation of motion. Note that $\vec{\pi}$ has an explicit time dependence. Express your result in terms of the electric and magnetic fields $\vec{E}$ and $\vec{B}$. Give a reason why the potentials $\vec{A}$ and A^0 must disappear from the final result.
(e) For this subproblem use the Dirac-Pauli representation of the Dirac matrices. Write

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix},$$

where φ and χ are 2-component Pauli spinors. Then rewrite $i\partial\psi/\partial t = H\psi$ as coupled equations for the Pauli spinors.

Problem 13

Review the quantization of the linear chain. Write

$$q_n(t) = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left(c_k(t)u_n^k + c_k^\dagger(t)u_n^{k\,*}\right) \text{ and } p_n(t) = -i\sum_k \sqrt{\frac{\hbar}{2m\omega_k}} \left(c_k(t)u_n^k - c_k^\dagger(t)u_n^{k\,*}\right)$$

for the set of conjugate field operators. Then express the Hamiltonian operator H for the quantum chain in terms of c_k and $c_k^\dagger$.

Problem 14

The complex Klein-Gordon field has a conserved Noether current

$$j^\mu(x) = -i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi)\,,$$

i.e. $\partial_\mu j^\mu = 0$, due to a global phase symmetry of the Lagrangian density.
(a) Write down the operator version $\hat{Q}$ of the conserved charge

$$Q = \int d^3x j^0(x)$$

in terms of the field operators $\hat{\phi}(x)$ and $\hat{\pi}(x)$.
(b) Derive an expression for $\hat{Q}$ solely in terms of the 'ladder' operators $\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}}^\dagger$ and $\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}}^\dagger$.

EXAMS AND MORE

- Homework will be collected, but not graded. The fractional return on the homework will be assigned a score $H = 0 \ldots 100$. Homework is due two weeks after assignment The deadline for homework submission is at the conclusion of the final exam.

- There will be one final exam. The length of each exam is the usual class period. Each exam score $E_i = 0 \ldots 100$, $i = 1$, carries equal weight. The class textbook only is allowed, and required. Other materials are a calculator, a ruler, paper and pen(cil).

- The overall score X for the course is determined by combining homework and exams as

$$X = \frac{4}{5}H + \frac{1}{5}\frac{E_1}{1}\,.$$

- Tentative exam schedule, Fall 2009
  Dec-07 (Mon)

- Announcements made in class may supersede syllabus rules!

- An approximate guideline for the grading scale is:

Points X	Grade
95–100	A
90–95	A
85–90	$A-$
80–85	$B+$
75–80	B
70–75	$B-$
65–70	$C+$
60–65	C
55–60	$C-$
50–55	$D+$
45–50	D
40–45	$D-$
0–40	F