

# Geometric Remarks on the Level Curves of Harmonic Functions

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## 1. Introduction

Many interesting open problems in Analysis concern the geometry of level sets of harmonic functions. In particular, we are concerned with the properties of  $Z$ , the zero set of a harmonic function  $u$ , and with connections to unique continuation problems. There seem to be very few geometric results about these zero sets, even for harmonic functions on  $R^2$ . Of course, if  $Z$  contains a simple closed curve, then  $u$  vanishes inside, by the maximum principle, and thus it vanishes on  $R^2$ . We will usually assume that  $u$  is nontrivial, unless stated otherwise.

In [F], the author shows that if a curve in  $Z$  is parametrized by  $x = p(t)$  and  $y = q(t)$ , where  $p(t)$  and  $q(t)$  are polynomials with real coefficients, then the curve is a straight line or a parabola. Therefore the zero set of an harmonic function can contain  $y = x^2$  but it cannot contain a cubic, such as  $y = x^3$ . See also [FNS]. These results do not apply, for example, to  $u(x, y) = xy + 1$ , since hyperbolas cannot be parametrized by polynomials.

It is not clear whether any simple geometric properties distinguish curves like  $y = x^3$  from those like  $y = x^2$ . It is certainly possible for  $Z$  to have an inflection point. For example, let  $u = x^3 - 3xy^2 + x + y$ . Then, the origin is an inflection point of  $Z$ .

This paper aims to show in various ways that  $Z$  cannot "wiggle very much". We investigate variations of the simple remark above, that  $Z$  cannot contain a simple closed curve, at least when  $u$  is harmonic, and this leads to some results involving area, but our main results concern the curvature  $\kappa$  of  $Z$ . It is impossible to bound  $\kappa$  independently of  $u$  and  $D$ ; an example with large curvature is given in Sec 2.2, but rescaling of almost any function will also produce a large  $\kappa$ . However, when  $Z$  is a *single* connected curve in  $D = B_1(0)$ , we can prove a fairly sharp bound on  $\kappa$ , at any point  $P \in Z$ . The bound depends only on the distance from  $P$  to the boundary. Our main result assumes for simplicity that  $P$  is the origin; it is proved in Section 2:

**Theorem 1.1** *Suppose that  $\Delta u = 0$  on  $B_r(0) = \{|x| \leq r\} \subset R^2$  with boundary values  $f(\theta)$ . Assume  $f(\theta) \geq 0$  on an interval  $I = [\alpha, \beta] \subset [-\pi, \pi]$  and  $f(\theta) \leq 0$  otherwise. Suppose  $u(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = 0$ . Then  $Z$  is a curve and its curvature at  $(0, 0)$  is bounded by  $\frac{C}{r}$ , where  $C \leq 24$  does not depend on  $u$ .*

If  $Z$  is a single curve in the entire plane (the case  $r = \infty$  of this theorem), we might heuristically conclude that  $\kappa = 0$  at every point of  $Z$ , so that  $Z$  and  $u$  must be linear. This claim is true, but we can prove only a special case of it directly from Theorem 1.1. Instead, we prove the general version in Section 3, using complex variables. Likewise, if  $Z$  consists of  $m$  curves,  $u$  must be a polynomial of degree  $m$ ; this is a slightly improved version of a theorem in [WWZ].

The paper [WWZ] is a good introduction to this one, since it clarifies the local structure of  $Z$ . Near each point  $P \in Z$ ,  $u$  behaves like a rotation of  $\text{Re}(z - P)^m$ , for some  $m \geq 1$ . So, locally,

$Z$  consists of  $m$  analytic curves passing through  $P$ . We can conclude that on any compact set,  $Z$  consists of finitely many maximal analytic curves, e.g., ones not contained in a larger analytic curve. In an open set  $D$ ,  $Z$  may split into infinitely many such curves. In fact, our Theorems 3.2 and 3.3 imply that this *must* happen when  $D = R^2$  and  $u$  is not a polynomial. It can also happen on  $D = B_1(0)$ , if  $u$  does not extend analytically to the boundary. For example, consider the harmonic function with boundary values  $\theta \sin\left(\frac{1}{\theta}\right)$ ,  $-\pi < \theta \leq \pi$ , which vanishes infinitely often on the boundary of  $D$ .

Section 3 also contains some other linearity results. For example, when  $Z$  contains a line segment, it contains a full line. Several of these theorems can be viewed as uniqueness results for harmonic functions, based on properties of  $Z$ . They partially answer a very general problem; suppose that  $u$  is harmonic on a domain  $D \subseteq R^n$  and  $u \equiv 0$  on a set  $Z \subseteq D$ . What properties of  $Z$  imply that  $u \equiv 0$  on all of  $D$ ? For example, if  $Z$  contains an open ball, then  $u$  vanishes on  $D$ ; this is often called the *unique continuation property* (UCP). Or, if  $Z$  is a single point, and  $u$  vanishes with all its derivatives there (we also say that  $u$  vanishes to infinite order at this point) then  $u$  vanishes on  $D$ ; this is the *strong unique continuation property* (SUCP). These properties extend to certain classes of nonharmonic functions discussed in some parts of this paper. Suppose

$$\Delta u(x) \leq V(x)|u(x)|, \tag{1.1}$$

where  $V$  is called the potential of  $u$ . If  $V \in L_{loc}^{\frac{n}{2}}(D)$  (where  $D \subseteq R^n$ ), then  $u$  has the UCP and SUCP<sup>1</sup> [JK]. There is an extensive literature on these problems; See for example the survey paper of T. Wolff [Wo] and a recent article of H. Koch and D. Tataru [KT] and the literature cited there.

Typically, unique continuation proofs depend on Carleman estimates. The authors attempted a new approach in [DH], and became curious about the intermediate case, in which  $Z$  is a set of positive dimension  $0 < d < n$ , rather than an open ball, or a single point.

Any harmonic function in  $R^2$  that vanishes on a closed curve must vanish also inside, by the maximum principle, but for general solutions of (1.1), there is no maximum principle, and  $Z$  may contain closed curves. However, we prove that the area of a region enclosed by  $Z$  cannot be arbitrarily small. We prove a lower bound that depends on  $\|V\|_{\infty}$ . We also include variations of this result for  $R^n$ ,  $n > 2$ .

In Section 5, we discuss a related conjecture: if the curve is "almost closed", then  $u$  must vanish. However, this idea seems very unlikely unless we impose fairly severe conditions on  $u$ , for example, that  $u$  is a harmonic polynomial. Section 5 contains several other open questions about extending subsets of  $Z$ , related to Section 3, and about the local arc length of  $Z$ .

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<sup>1</sup>In this context, that  $u \in W^{2,p}(D)$  vanishes to infinite order at  $x_0$  if for every  $N > 0$ ,

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-N} \int_{|x-x_0| < \epsilon} |u(x)|^2 dx = 0.$$

## Section 2.

In this Section, we investigate the curvature  $\kappa$  of  $Z \subset \mathbf{R}^2$ , when  $u$  is harmonic. R. Jerrard and L. Rubel proved in [JR] that  $\log |\kappa|$  is superharmonic, on a nice enough domain. See [Wa] for another monotonicity property of curvature. Unfortunately, these do not seem to say much about a given level curve. Before proving Theorem 1.1 and some corollaries, let's recall a few useful curvature formulas.

### 2.1 Curvature in $\mathbf{R}^2$

The standard formula for the curvature of the graph of a function  $y = f(x) \in C^2(\mathbf{R})$  at a point  $P = (p, f(p))$  is

$$\kappa f(P) = \frac{f''(p)}{(1 + [f'(p)]^2)^{\frac{3}{2}}}.$$

We use this formula to compute the curvature of the level sets of a  $C^2(D)$  function  $u(x, y)$  at points where the gradient does not vanish. Here  $D$  is an open set of  $\mathbf{R}^2$ .

Let  $L = u^{-1}(c)$  be the set of points where  $u \equiv c$ . Let  $P = (p, q) \in L$  be such that  $\nabla u(P) \neq 0$ . Suppose that  $L$  is the graph, in a neighborhood of  $P$ , of a function  $y = y(x)$ . By the implicit function theorem, there exists a neighborhood of  $P$  in  $L$  in which

$$y'(x) = -\frac{u_x(x, y(x))}{u_y(x, y(x))}.$$

Consequently,

$$\begin{aligned} y''(P) &= -\frac{(u_{xx}(P) + y'(p)u_{xy}(P))u_y(P) - u_x(P)(u_{xy}(P) + y'(p)u_{yy}(P))}{u_y(P)^2} \\ &= -\frac{u_y^2(P)u_{xx}(P) - 2u_{xy}(P)u_x(P)u_y(P) + u_x^2(P)u_{yy}(P)}{u_y(P)^3}. \end{aligned}$$

The curvature of  $L$  at  $p$  is

$$\kappa u(P) = \frac{y''(p)}{(1 + (y')^2(p))^{\frac{3}{2}}},$$

and by the previous calculation,

$$\kappa u(P) = -\frac{u_y^2(P)u_{xx}(P) - 2u_{xy}(P)u_x(P)u_y(P) + u_x^2(P)u_{yy}(P)}{(u_x^2(P) + u_y^2(P))^{\frac{3}{2}}}. \quad (2.1)$$

So,  $\kappa u(P)$  associates to each  $P \in D$  (with  $\nabla u(P) \neq (0, 0)$ ), the curvature of the level set of  $u$  that passes through  $P$ . We say in short that  $\kappa u(P)$  is *the curvature of  $u$  at  $P$* . If we let

$$\tilde{H}u(P) = \begin{pmatrix} -u_{yy}(P) & u_{xy}(P) \\ u_{xy}(P) & -u_{xx}(P) \end{pmatrix}, \text{ then}$$

$$\kappa u(P) = |\nabla u(P)|^{-3} \langle \nabla u(P), \tilde{H}u(P)\nabla u(P) \rangle. \quad (2.2)$$

Note that if  $u$  is harmonic, then the matrix  $\tilde{H}u(P)$  is the Hessian matrix of  $u$  at  $P$  because  $u_{xx}(P) = -u_{yy}(P)$ .

Let us recall one more useful curvature formula. If  $u = \operatorname{Re}(w)$ , where  $w$  is a holomorphic function in  $D$ , then the curvature of a level set of  $u$  at point  $P$  where  $\nabla u(P) \neq 0$  is

$$\kappa w(P) = |w'(P)| \operatorname{Re} \left( \frac{w''(P)}{(w'(P))^2} \right). \quad (2.3)$$

See e.g. [JR].

## 2.2 Proof of Theorem 1.1

**Proof:** We can assume  $r = 1$  by scaling. Let  $u(x, y)$  be harmonic on the unit disk, with  $u(0, 0) = 0$ . We can assume  $\nabla u(0, 0) \neq \mathbf{0}$ ; otherwise,  $Z$  would intersect itself there, and therefore intersect the unit circle more than twice (by the maximum principle). By rotation, we can assume  $u_y(0, 0) = 0$ , and that  $D = \pi u_x(0, 0) > 0$ . Then  $u$  can be represented by the Poisson integral in terms of its boundary values,  $f(\theta)$ ,

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{(1 - x^2 - y^2) d\theta}{(x - \cos \theta)^2 + (y - \sin \theta)^2} \quad (2.4)$$

By hypothesis,  $f > 0$  on an interval  $I = [a, b] \subset [0, 2\pi]$ . Without loss of generality,  $\int_I f(\theta) d\theta = 1$ . We have  $\int_0^{2\pi} f(\theta) d\theta = u(0, 0) = 0$ . Several averages, especially ones involving  $\cos \theta$ , reappear throughout this proof. We denote by  $D$  and  $C$  those taken over  $[0, 2\pi]$  and  $I$ , resp. We use  $d$  and  $c$  for the corresponding ones with  $\cos^2 \theta$ . Define

$$S = \int_I f(\theta) \sin \theta d\theta, \quad C = \int_I f(\theta) \cos \theta d\theta, \quad c = \int_I f(\theta) \cos^2 \theta d\theta,$$

$$D = \pi u_x(0, 0) = \int_0^{2\pi} f(\theta) \cos \theta d\theta, \quad d = -\frac{\pi}{4} u_{yy}(0, 0) = \int_0^{2\pi} f(\theta) \cos^2 \theta d\theta.$$

$$\text{And note } 0 = u_y(0, 0) = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin \theta d\theta.$$

From (2.1),  $|\kappa| = \left| \frac{u_{yy}}{u_x} \right| = \left| \frac{4d}{D} \right|$ . We can assume  $d > 0$  (if not, replace  $u$  by  $-u$ , rotate by  $\pi$  radians, and replace  $[a, b]$  by its complement). So, we must prove  $d \leq 6D$ .

Roughly speaking, this means we need to bound  $\frac{c}{C}$ . We will prove a bound like this for the positive part of  $u$ , defined from  $f$  restricted to  $[a, b]$ , and for the negative part separately. We will define  $G_0$  precisely below, but it is basically the contribution to  $D$  from an extreme  $f$ , consisting of point masses at  $a$  and  $b$ . Likewise,  $g_0$  can be thought of as the contribution to  $c$  (and  $d$ ) from the same extreme function. We will compare our  $f$  to the extreme case by proving Lemma 2.1 below.

We will now define  $g_0$  and  $G_0$ . Given two angles, such that  $\sin(\theta) > S > \sin(\gamma)$ , there is a unique  $\alpha = \alpha(\theta, \gamma) \in (0, 1)$ , such that  $\alpha \sin(\theta) + (1 - \alpha) \sin(\gamma) = S$ , namely  $\alpha = \frac{S - \sin(\gamma)}{\sin(\theta) - \sin(\gamma)}$ . Set  $G(\theta, \gamma) = \alpha \cos(\theta) + (1 - \alpha) \cos(\gamma)$ , which is just the  $x$  coordinate of the intersection of the line  $y = S$  with the line through  $e^{i\theta}$  and  $e^{i\gamma}$ . According to Lemma 2.3 below,  $\sin(b) > S > \sin(a)$ , so we can define  $G_0 = G(b, a)$ . Likewise, set  $g(\theta, \gamma) = \alpha \cos^2(\theta) + (1 - \alpha) \cos^2(\gamma)$  and let  $g_0 = g(b, a)$ . The proof of Theorem 1.1 is a consequence of the following

**Lemma 2.1**

$$|c - g_0| \leq 6(C - G_0). \quad (2.5)$$

To see the plan of the proof, suppose  $f|_I$  is concentrated at the endpoints of  $I$ . Then both sides of (2.5) are zero. The claim is that if we smooth out this  $f$ , we increase  $C$  as fast as  $c$ . To make this plan work, we need to define and relate some parameters; an angle  $\theta$  near  $b$ , a lower angle  $\gamma$  near  $a$ , and  $t \in [0, 1]$ .

Since  $\sin(a) < \sin(b)$ , there is a  $p \in (a, b)$  such that  $\sin(p) = S$ . Likewise, there is a  $p_2 \in (b, a + 2\pi)$  such that  $\sin(p_2) = S$ . Since  $\sin(x) = S$  has only two solutions with  $x \in (a, a + 2\pi)$ ,  $p$  and  $p_2$  are unique. So, for  $a < \gamma < p < \theta < b$ , we can define  $T(\theta, \gamma) = \frac{f(\theta) \sin \theta - S}{f(\gamma)(S - \sin \gamma)} > 0$ . Define  $\gamma = \gamma(\theta)$  by  $d\gamma/d\theta = -T(\theta, \gamma) < 0$ , with  $\gamma(b) = a$ .

The domain and range of  $\gamma$  are not yet clear, since the construction requires  $\gamma < p$ . But we claim it defines a bijection  $\gamma : (p, b] \rightarrow [a, p)$ . As long as  $\gamma < p < \theta$ , we have

$$\int_a^\gamma f(\gamma)(S - \sin \gamma) d\gamma = \int_\theta^b f(\theta)(\sin \theta - S) d\theta$$

and from the definition of  $S$ ,

$$\int_a^b f(\theta)(\sin \theta - S) d\theta = 0$$

If either  $\gamma = p$  or  $\theta = p$ , these equations imply that  $\gamma = \theta = p$ , which proves our claim about  $\gamma$ . Also, we can define  $\gamma(p) = p$ , with continuity.

We now regard  $\alpha$ ,  $G$  and  $g$  as functions of  $\theta$  alone. Yet another parameter will be useful.

Define  $t = t(\theta)$  by  $\frac{dt}{d\theta} = -\frac{1}{\alpha}f(\theta)$ , with  $t(b) = 0$ , which implies  $\frac{dt}{d\gamma} = \frac{f(\gamma)}{1 - \alpha}$ . Then

$$\int_p^b f d\theta = \int_0^{t(p)} \alpha dt \quad \text{and} \quad \int_a^p f d\gamma = \int_0^{t(p)} (1 - \alpha) dt.$$

Now, add and get  $1 = \int_0^{t(p)} dt = t(p)$ , so  $t : [p, b] \rightarrow [0, 1]$ . Likewise, we can multiply the integrands by  $\cos(\theta)$  in the first equation and by  $\cos(\gamma)$  in the second, to get

$$C = \int_p^b f(\theta) \cos(\theta) d\theta + \int_a^p f(\gamma) \cos(\gamma) d\gamma = \int_0^1 G(\theta(t)) dt$$

The same procedure with  $\cos^2$  leads to

$$c = \int_0^1 g(\theta(t)) dt$$

Recall that  $G_0 = G(b, a) = G(\theta(0))$  and  $g_0 = g(b, a) = g(\theta(0))$ . We will prove that  $\frac{\partial G}{\partial \theta} < 0$  and  $\frac{\partial G}{\partial \gamma} > 0$ , so  $\frac{dG}{d\theta} < 0$  and  $G(\theta(t))$  increases with  $t$ . So,  $C > G_0$ . To prove Lemma 2.1, we must bound

$$\frac{|c - g_0|}{C - G_0} = \frac{|\int_0^1 g dt - g_0|}{\int_0^1 G dt - G_0} = \frac{|g(\theta) - g_0|}{G(\theta) - G_0}$$

for some  $\theta \in (p, b)$ , by the Mean Value Theorem. We must show

$$|g(\theta) - g_0| \leq 6(G(\theta) - G_0). \quad (2.6)$$

**Lemma 2.2**

$$\left| \frac{\partial g}{\partial \theta} \right| \leq -6 \frac{\partial G}{\partial \theta} \quad \text{and} \quad \left| \frac{\partial g}{\partial \gamma} \right| \leq 6 \frac{\partial G}{\partial \gamma}. \quad (2.7)$$

Assuming this for the moment, we can prove (2.6), and Lemma 2.1. Indeed

$$\left| \frac{dg}{d\theta} \right| \leq \left| \frac{\partial g}{\partial \theta} \right| + \left| \frac{\partial g}{\partial \gamma} \frac{d\gamma}{d\theta} \right| \leq -6 \frac{\partial G}{\partial \theta} - 6 \frac{\partial G}{\partial \gamma} \frac{d\gamma}{d\theta} = -6 \frac{dG}{d\theta}.$$

*Proof of Lemma 2.2:* From the definition of  $\alpha$ ,  $\frac{\partial \alpha}{\partial \theta} = -\alpha \frac{\cos(\theta)}{\Delta_s}$ , where  $\Delta_s = \sin(\theta) - \sin(\gamma) > 0$ . Likewise, set  $\Delta_c = \cos(\theta) - \cos(\gamma)$ , and  $\Delta_{c^2} = \cos^2(\theta) - \cos^2(\gamma)$ . So,

$$\frac{\partial G}{\partial \theta} = \frac{\partial \alpha}{\partial \theta} \Delta_c - \alpha \sin(\theta) \quad \text{and}$$

$$-\frac{\partial G}{\partial \theta} \frac{\Delta_s}{\alpha} = \cos(\theta) \Delta_c + \sin(\theta) \Delta_s = \mathbf{w}_\theta \cdot (\mathbf{w}_\theta - \mathbf{w}_\gamma) > \frac{\Delta_c^2}{4} \quad (2.8)$$

where  $\mathbf{w}_\theta$  is the vector  $(\cos \theta, \sin \theta) \in \mathbf{R}^2$ , and the dot is the usual scalar product in  $\mathbf{R}^2$ . For the last step above, see Lemma 2.4. Note that  $\frac{\partial G}{\partial \theta} < 0$ . Likewise,

$$\begin{aligned} -\frac{\partial g}{\partial \theta} \frac{\Delta_s}{\alpha} &= \cos \theta \Delta_{c^2} + 2 \sin \theta \cos \theta \Delta_s \\ &= 2 \cos \theta [\cos \theta \Delta_c + \sin \theta \Delta_s] - \cos \theta \Delta_c^2 \\ &= 2 \cos \theta \left[ -\frac{\partial G}{\partial \theta} \frac{\Delta_s}{\alpha} \right] - \cos \theta \Delta_c^2 \end{aligned}$$

With (2.8) and the triangle inequality, this proves the first part of Lemma 2.2.

A similar calculation proves  $\left| \frac{\partial g}{\partial \gamma} \right| \leq 6 \frac{\partial G}{\partial \gamma}$ . Set  $\beta = 1 - \alpha$ . Then  $\beta' = \frac{\partial \beta}{\partial \gamma} = \frac{\beta \cos \gamma}{\Delta_s}$

$$\frac{\Delta_s}{\beta} \frac{\partial G}{\partial \gamma} = -(\cos \gamma \Delta_c + \sin \gamma \Delta_s) = \mathbf{w}_\theta \cdot (\mathbf{w}_\theta - \mathbf{w}_\gamma) > \frac{\Delta_c^2}{4}$$

much as before. And

$$\begin{aligned} \frac{\partial g}{\partial \theta} \frac{\Delta_s}{\beta} &= 2 \cos \gamma [\cos \gamma \Delta_c + \sin \gamma \Delta_s] + \cos \gamma [\Delta_c]^2 \\ &= 2 \cos \gamma \left[ -\frac{\partial G}{\partial \gamma} \frac{\Delta_s}{\beta} \right] + \cos \gamma \Delta_c^2 \end{aligned}$$

which shows  $\left| \frac{\partial g}{\partial \gamma} \right| \leq 6 \frac{\partial G}{\partial \gamma}$ , as claimed.

This settles Lemma 2.1, which was about the derivatives of the positive harmonic function  $u_+$  with boundary values  $f \cdot \chi_I$ . Set  $u_- = u_+ - u \geq 0$ , which has boundary values  $-f \cdot \chi_{I^c} \geq 0$ .

Set  $v(x, y) = u_-(-x, y)$ . Its boundary values are simply a reflection of those of  $u_-$ , non-negative on  $(a_v, b_v)$ , where  $\sin(a_v) = \sin(a)$  and  $\cos(a_v) = -\cos(a)$ , with similar relations for  $b_v$  and  $b$ . We can apply Lemma 2.1 to  $v$  instead of  $u_+$ , with no change to the values of  $S$  or  $\alpha(b, a)$  or  $g_0$ . But we must replace  $G_0$  by  $-G_0$ ,  $c$  by  $c - d$ , and  $C$  by  $D - C$ . Lemmas 2.3 and 2.4 still hold. This gives

$$|(c - d) - g_0| \leq 6(D - C + G_0) \quad (2.9)$$

The triangle inequality, with Lemma 2.1 and (2.9), shows  $d \leq |c - g_0| + |(c - d) - g_0| \leq 6D$ , which proves the Theorem.

**Lemma 2.3** *With  $S$ ,  $a$  and  $b$  as in Theorem 1.1,  $\sin(b) > S > \sin(a)$ .*

**Proof:** Let  $m$  be the midpoint of  $(a, b)$  and let  $\mathbf{w}_m$  be the unit vector  $(\cos(m), \sin(m)) \in \mathbf{R}^2$ . For every  $\theta \in (a, b)$ ,  $\mathbf{w}_\theta \cdot \mathbf{w}_m > \mathbf{w}_b \cdot \mathbf{w}_m = \cos(m - b) = \nu$ . Recall that  $u = u_+ - u_-$  is the difference of two positive harmonic functions, which satisfy formulas similar to the ones for  $u$ . So,  $\pi \nabla u_+(0, 0) = (C, S)$  is an average of vectors of the form  $\mathbf{w}_\theta$ , so  $\pi \nabla u_+(0, 0) \cdot \mathbf{w}_m > \nu$ . Likewise,  $\pi \nabla u_-(0, 0) \cdot \mathbf{w}_m < \nu$ . So  $0 < \pi \nabla u(0, 0) \cdot \mathbf{w}_m = D \cos(m)$ , and  $\cos(m) > 0$ . So,  $\sin(b) - \sin(a) = \sin(m + (b - m)) - \sin(m - (b - m)) = 2 \cos(m) \sin(b - m) > 0$ , since  $0 < b - m < \pi$ . So,  $\sin(b) > \sin(a)$ .

$S$  is a weighted average of  $\sin(\theta)$  over  $a < \theta < b$ , so it is less than the maximum value of this function. The maximum is 1 if and only if  $\frac{\pi}{2} \in (a, b)$ ; otherwise it is  $\sin(b)$ . Since  $u_y(0, 0) = 0$ ,  $S$  is an average of  $\sin(\theta)$  over  $(b, a + 2\pi)$  as well, with similar conclusions about the max. But  $\frac{\pi}{2}$  can't belong to both intervals, so in one case the maximum is  $\sin(b)$ . So,  $S < \sin(b)$ . By similar reasoning,  $S > \sin(a)$ .

**Lemma 2.4** *For any two angles  $\gamma \neq \theta$ ,  $\Delta_c^2 < 4\mathbf{w}_\theta \cdot (\mathbf{w}_\theta - \mathbf{w}_\gamma)$ .*

**Proof:** Letting  $\mathbf{v} = \mathbf{w}_\theta - \mathbf{w}_\gamma = (\Delta_c, \Delta_s)$ , we see  $\Delta_c^2 \leq \|\mathbf{v}\|^2$ . Now, suppose  $\beta$  is the angle between  $\mathbf{w}_\theta$  and  $\mathbf{v}$ , and  $\nu \in (0, \pi)$  is the angle between  $\mathbf{w}_\theta$  and  $\mathbf{w}_\gamma$ . So,  $2\beta + \nu = \pi$ . Also,  $\|\mathbf{v}\| < \nu < 4 \sin(\nu/2)$  (since  $\nu < \pi$ ). And  $\mathbf{w}_\theta \cdot (\mathbf{w}_\theta - \mathbf{w}_\gamma) = \|\mathbf{v}\| \cos(\beta) = \|\mathbf{v}\| \sin(\nu/2) > \frac{\|\mathbf{v}\|^2}{4}$  as desired.

### 2.3 Corollaries of Theorem 1.1.

The previous theorem controls curvature only at the center of the ball. Next, we will study the curvature of  $Z$  at a general point  $a \in Z \cap B_1(0)$  via conformal mapping. But the bound must grow as  $|a| \rightarrow 1$ . To see this, recall the example  $u$  given at the beginning of Section 2.2. There,  $Z$  is a hyperbola centered at  $(\epsilon, \epsilon)$  with very large curvature,  $\kappa u(p) = \frac{1}{\epsilon\sqrt{2}}$  at the origin. We can translate this function so that the new hyperbola is centered on the circle, at  $(\sqrt{1/2}, \sqrt{1/2})$ . This new  $Z$  crosses the circle only twice, thus satisfying the assumptions of the Theorem. It has the same large curvature, but at a new point  $a$ , with  $1 - |a| = \epsilon/\sqrt{2} = \frac{1}{2\kappa}$ . The following Corollary gives the best possible bound on the curvature at a general point  $a$  in the ball, in terms of the best constant  $C$  in Theorem 1.1 (but  $C$  is not currently known).

**Corollary 2.5** *Let  $u$  be a harmonic function in the unit disk. Suppose that  $Z$  is a single connected curve, as in Theorem 1.1. Then, for every  $a \in Z$ , the curvature  $\kappa(u)(a)$  of  $Z$  at  $a$  satisfies the following inequality.*

$$|\kappa u(a)| \leq \frac{C + 2|a|}{1 - |a|^2} \leq \frac{C_2}{1 - |a|}. \quad (2.10)$$

where  $C$  is the constant in Theorem 1.1 and  $C_2$  is some constant independent of both  $a$  and  $u$ .

*Proof.* Let  $a \in Z$  be such that  $u(a) = 0$ . Assume  $u = \operatorname{Re}(w)$ , where  $w$  is a holomorphic function in the unit disk. Consider the conformal transformation  $g_a(z) = \frac{z+a}{1-\bar{a}z}$ ,  $z = x + iy$ . This transformation maps the unit disk and the unit circle onto themselves. The function  $w_a(x, y) = w \circ g_a(x, y)$  is holomorphic and its real part,  $u_a(x, y) = u \circ g_a(x, y)$ , is harmonic. By the formula (2.3), the curvature of the zero set of  $u_a$  at the origin is

$$\begin{aligned} \kappa u_a(0) &= |w'_a(0)| \operatorname{Re} \left( \frac{w''_a(0)}{(w'_a(0))^2} \right) \\ &= |w'(a)|(1 - |a|^2) \operatorname{Re} \left( \frac{(g'_a(0))^2 w''(a) + w'(a) g''_a(0)}{(g'_a(0) w'(a))^2} \right) \\ &= |w'(a)|(1 - |a|^2) \left( \operatorname{Re} \left( \frac{w''(a)}{(w'(a))^2} \right) + \operatorname{Re} \left( \frac{g''_a(0)}{w'(a) (g'_a(0))^2} \right) \right) \\ &= (1 - |a|^2) \kappa(u)(a) - 2|w'(a)| \operatorname{Re} \left( \frac{\bar{a}}{w'(a)} \right) \\ &= (1 - |a|^2) \kappa(u)(a) - 2 \operatorname{Re} \left( \frac{\bar{a}|w'(a)|}{w'(a)} \right). \end{aligned}$$

We have used the fact that  $g'_a(z) = \frac{1 - |a|^2}{(1 + z\bar{a})^2}$  and  $g''_a(z) = \frac{-2\bar{a}(1 - |a|^2)}{(1 + z\bar{a})^3}$ .

Since  $\left| \operatorname{Re} \left( \frac{\bar{a}|w'(a)|}{w'(a)} \right) \right| < |a|$ , we have the following inequalities.

$$\kappa u(a) = (1 - |a|^2)^{-1} \left( \kappa u_a(0) + 2 \operatorname{Re} \left( \frac{\bar{a}|w'(a)|}{w'(a)} \right) \right) \leq (1 - |a|^2)^{-1} (\kappa u_a(0) + 2|a|),$$

and

$$\kappa u(a) \geq (1 - |a|^2)^{-1} (\kappa u_a(0) - 2|a|).$$

Since by Theorem 1.1,  $|\kappa u_a(0)| < C$ , we have proved (2.10).

### Section 3

We now consider some conditions on  $Z$ , which imply that  $u$  must be linear. This section also includes results about polynomials, and on the linearity of  $Z$ . Recall that Theorem 1.1 controls the curvature of  $Z$  inside a ball in the plane, if  $Z$  cuts the boundary only twice. That proof does not seem to extend directly to the entire plane, but a slightly stronger result can be proved in that case using complex analytic methods.

**Theorem 3.1** *Let  $u$  be harmonic on the plane, and assume  $U_+ = \{z : u(z) > 0\}$  and  $U_- = \{z : u(z) < 0\}$  are both connected. Then  $u$  is linear.*

**Proof:** The authors' original proof, using mostly real analysis, resembles that of Theorem 5 in [WWZ]. We present here an elegant complex analytic version, found independently by Bao Qin Li [L]. Let  $f = v + iu$  be analytic on the plane. The zeroes of  $f$  all lie in  $Z$ . Suppose  $n$  zeroes lie inside a simple closed curve  $\Gamma$ , which intersects  $Z$  just twice. So,  $f(\Gamma)$  winds around the origin at most once. By the Argument Principle,  $n \leq 1$ . So,  $f$  has at most one zero in the plane. The same is true for  $f - a$ , for any  $a \in \mathbf{R}$ . So,  $f$  must be a polynomial (otherwise,  $f(z) = a$  would occur infinitely often, for most values of  $a$ ). Since  $f$  has only one zero (counting multiplicity), it is linear.

The following two theorems generalize Theorem 3.1 and are roughly equivalent to each other.

**Theorem 3.2** *If  $U_+$  and  $U_-$  have a total of  $n$  components, then  $u$  is a polynomial of degree  $m$ , with  $\frac{n}{2} \leq m \leq n - 1$ .*

**Theorem 3.3** *If  $Z$  is a finite union of  $m$  maximal analytic curves, then  $u$  is a polynomial of degree  $m$ .*

**Proof of Theorem 3.3.** Theorem 5 of [WWZ] shows that  $u$  is a polynomial of degree  $d < \infty$ , but it doesn't bound  $d$ . Nor does the proof of Theorem 3.2, without additional work. But outside some large ball  $B_R(0, 0)$ , the highest order term of  $u$  dominates, and it behaves like  $\operatorname{Re}(z^d)$ . So, outside  $B_R$ ,  $Z$  consists of exactly  $2d$  curves extending to infinity. But [WWZ] shows that each maximal analytic curve in  $Z$  extends to infinity exactly twice. So,  $2m = 2d$  and  $d = m$ .

**Proof of Theorem 3.2.** Euler's Theorem for planar graphs states that in a planar graph the numbers of vertices  $v$ , faces  $f$  and edges  $e$  satisfy  $v - e + f = 2$ . This implies that  $n = m + v + 1$  where  $m$  is the number of intersections of the analytic curves, by multiplicity. The intersections can occur only at roots of  $f'(z)$  on  $Z$ , so  $0 \leq v \leq m - 1$  and the theorem follows.

It is interesting to speculate whether this kind of result generalizes to other domains. Suppose  $u$  is harmonic on the sphere  $S_2$  (the boundary of  $B_1(0) \subset \mathbf{R}^3$ ), except possibly at the point  $(0,0,1)$ . Assume  $U_+$  and  $U_-$  have only one component each. Then we can map the sphere to the plane conformally (for example by the stereographic projection  $f(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ , so that  $(0,0,1)$  maps to infinity). Then  $u \circ f^{-1}$  is harmonic on the plane, and  $f(Z)$  splits the plane in two. So, we can conclude that  $f(Z)$  is a straight line, and  $Z$  is a circle in  $S_2$  through  $(0,0,1)$  (and so is every other level curve of  $u$ ).

There seems to be no obvious generalization of these three theorems if  $u$  is harmonic on  $\mathbf{R}^3$  instead of  $R^2$ . Here is a simple counterexample to a generalized Theorem 3.1. Let  $u(x, y, z) = xy + z$ , which is harmonic, but not linear, on  $\mathbf{R}^3$ . We claim that every point in  $U_+$  is connected to  $P = (0, 0, 1)$  by a path in  $U_+$ , so this set is connected. To see this, let  $U_a = U_+ \cap \{z = a\}$ . If  $a > 0$  then  $U_a$  is connected, and the union of these is also connected (it contains the positive  $z$ -axis). If  $a \leq 0$ , then each  $U_a$  has two components. One contains the point  $(1 - a, 1 - a, a)$  (since  $(1 - a)^2 + a > 0$  for  $a \leq 1$ ) and the other contains  $(a - 1, a - 1, a)$ . These union of these points form two lines, contained in  $U_+$  (at least while  $z \leq 1$ ), which intersect at  $P$ . So,  $U_+$  is

path-connected via  $(0,0,1)$ .  $U_-$  is the image of  $U_+$  under the rotation  $(x, y, z) \rightarrow (-x, y, -z)$ , so it is also connected. Essentially the same example works in  $\mathbf{R}^4$ , etc.

**Theorem 3.4** *Suppose  $L$  is an analytic curve parametrized by  $\phi : \mathbf{R}^1 \rightarrow \mathbf{C}$ , that  $v$  is harmonic on some neighborhood of the curve, and that  $v(\phi(t)) = 0$  on some interval,  $a < t < b$ . Then  $v = 0$  on  $L$ .*

This result may follow from the theory of maximal analytic curves in [WWZ], but we provide an independent proof. It is enough to show that  $v(\phi(t))$  is real analytic at an arbitrary point, such as  $t = 0$ . We may assume  $\phi(0) = (0, 0)$ . We can express  $\phi(t)$  as a complex-valued series which converges uniformly for  $|t| < \epsilon$ .

$$\phi(t) = \sum_{j=1}^{\infty} (b_j + ic_j)t^j$$

Now replace  $t$  by a complex variable  $z$ . The radius of convergence of the series is at least  $\epsilon$ , so this formula extends  $\phi$  to an analytic function  $\Phi$  defined on a small ball centered at the origin. So,  $v \circ \Phi$  is harmonic, and therefore real analytic, with a power series expansion in  $s$  and  $t$ . When restricted to  $s = 0$ , it is still a real analytic function of  $t$ , namely  $v \circ \phi$ .

This has some simple corollaries, probably not new. The first could also be proved from the Schwartz reflection principle. The second is similar to a UCP result. It shows that a nontrivial harmonic function on the plane cannot vanish on an arc of a circle. It follows from the theorem and the maximum principle.

**Corollary 3.5** *Suppose that  $\Delta v = 0$  on the plane  $\mathbf{C}$ , and  $v = 0$  on some line segment  $[a, a+\epsilon] \subset \mathbf{R}^1 \subset \mathbf{C} = \mathbf{R}^2$ . Then  $v = 0$  on  $\mathbf{R}^1$ .*

**Corollary 3.6** *Suppose that  $\Delta v = 0$  on  $\mathbf{C} = \mathbf{R}^2$  and  $v = 0$  on the semicircle  $S = \{z \in \mathbf{C} : |2z - \mathbf{i}| = 1 \text{ and } \text{Im}(z) \geq 1/2\}$ . Then  $v = 0$  on  $\mathbf{C}$ .*

Of course,  $S$  could be replaced by any arc of the circle, with the same result. But if  $v$  fails to be harmonic at some point on the circle, the result does not hold (consider the Poisson kernel for the disk). If  $v$  is harmonic only on a neighborhood of the circle,  $v$  must vanish on the circle, but not necessarily at other points. For example,  $v = \ln(|2z - i|)$  vanishes on the circle, but it has a pole at the center. Likewise, 3.5 implies  $u = 0$  on  $R^1$  but not necessarily on  $R^2$  (consider  $u(x, y) = xy$ ). We mention some related open questions in Section 5.

## Section 4.

In this section we investigate the properties of the solutions <sup>2</sup> of the differential inequality

$$\Delta u(x) \leq V(x)|u(x)|, \quad x \in D \subset \mathbf{R}^n. \quad (4.1)$$

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<sup>2</sup>By solution we mean:  $u \in W^{2,p}(D)$  that satisfy (4.1) almost everywhere

where  $V(x) \in L^1_{loc}(D)$ . We assume that  $u$  vanishes on the boundary of  $D$ . Of course harmonic functions satisfy (4.1), and if they vanish on the boundary of  $D$  then they vanish everywhere. This property is not generally true for nonharmonic solutions, but we can prove it under suitable assumptions on the size of  $V$  and  $|D|$ . Major tools in our proofs are sharp Sobolev inequalities. The standard Sobolev inequality on a domain  $D \subset \mathbf{R}^n$

$$\|u\|_{\frac{np}{n-p}} \leq c(p, D) \|\nabla u\|_p \quad u \in W_0^{1,p}(D) \quad (4.2)$$

is valid for every  $p < n$ . It is easy to prove that this inequality is translation and scaling invariant. When  $p = 1$  and  $D$  is a ball in  $\mathbf{R}^n$ , the best possible constant  $c(p, D)$  is achieved by the characteristic function of the ball. It is

$$c(1, B_1(0)) = (n|\omega_n|^{\frac{1}{n}})^{-1} \quad (4.3)$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbf{R}^n$ . In particular, when  $n = 2$ ,  $c(1, B_1(0)) = (2\pi^{\frac{1}{2}})^{-1}$ . Using rearrangements, it is possible to prove that  $c(p, D) \leq c(p, B_1(0))$  for every  $D \subset \mathbf{R}^n$  such that  $|D| = |B_1(0)|$  and all  $1 \leq p < n$ . When  $p \neq 1$ ,

$$c(p, B_1(0)) = p \frac{p-1}{n-p}. \quad (4.4)$$

For the proof of these inequality see e.g [LL].

Since (4.2) is scaling-invariant,  $c(p, \mathbf{R}^n)$  is still the best constant when the functions are restricted to have their support in a fixed open set. We use the inequalities (4.2) and (4.4) to estimate the size of the support of the solutions of the differential inequality (4.1)

**Theorem 4.1** *Let  $u(x) \in W_0^{2,p}(D)$ , with  $D$  a domain of  $\mathbf{R}^n$ , be a nontrivial solution of the differential inequality (4.1). Assume  $u(x) > 0$  in  $D$  and  $u \equiv 0$  on the boundary of  $D$ .*

*i) If  $n = 2$  and  $|V(x)|$  is bounded in  $D$ , then*

$$|D| \geq 4\pi \|V\|_{\infty}^{-1}. \quad (4.5)$$

*ii) If  $n > 2$  and  $V \in L^r(D)$  with  $r > \frac{n}{2}$ , then*

$$|D| \geq \left( \frac{n-2}{2\|V\|_{\frac{1}{r}}} \right)^{\frac{2nr}{2r-n}}. \quad (4.6)$$

*iii) If  $V \in L^{\frac{n}{2}}(D)$ , with  $n > 2$ , then necessarily*

$$\|V\|_{\frac{n}{2}} \geq \frac{(n-2)^2}{4}. \quad (4.7)$$

*Remark.* We noted in the Introduction that the differential inequality (4.1) has the SUCP, (unique continuation from a point) and hence also the UCP, (unique continuation from an open set) whenever  $V \in L^{\frac{n}{2}}_{loc}(D)$ . Theorem 4.1 can be interpreted as a unique continuation theorem

from the boundary of the domain of  $u$ . That is, if  $u$  vanishes on the boundary of its domain of definition and the  $L^{\frac{n}{2}}$  norm of  $V$  is sufficiently small, then  $u \equiv 0$ .

*Proof of i), ( $n = 2$ ).* By the standard Sobolev inequality (4.3), (with  $p = 1$  and  $n = 2$ ), Hölder's inequality, (4.1), and the identity

$$\int_D |\nabla u(x)|^2 dx = \int_D u(x) \Delta u(x) dx, \quad u(x) \in W_0^{2,p}(D),$$

we get the following chain of inequalities:

$$\begin{aligned} \|u\|_2 &\leq (2\pi^{\frac{1}{2}})^{-1} \|\nabla u\|_1 \\ &\leq (2\pi^{\frac{1}{2}})^{-1} |D|^{\frac{1}{2}} \|\nabla u\|_2 \\ &\leq (2\pi^{\frac{1}{2}})^{-1} |D|^{\frac{1}{2}} \left( \int_D u(x) \Delta u(x) dx \right)^{\frac{1}{2}} \\ &\leq (2\pi^{\frac{1}{2}})^{-1} |D|^{\frac{1}{2}} \left( \int_D u^2(x) V(x) dx \right)^{\frac{1}{2}} \\ &\leq (2\pi^{\frac{1}{2}})^{-1} \|V\|_{\infty}^{\frac{1}{2}} |D|^{\frac{1}{2}} \|u\|_2. \end{aligned}$$

Since  $u \not\equiv 0$ ,  $(2\pi^{\frac{1}{2}})^{-1} \|V\|_{\infty}^{\frac{1}{2}} |D|^{\frac{1}{2}} \geq 1$ , which implies  $|D| \|V\|_{\infty} \geq 4\pi$ .

*Proof of ii) and iii), ( $n > 2$ ).* Assume  $V \in L^{\frac{n}{2}}(D)$ . We apply the sharp Sobolev inequality (4.4) with  $p = 2$ , and the tools that we have used in the first part of the proof. We obtain

$$\begin{aligned} \|u\|_{\frac{2n}{n-2}} &\leq \frac{2}{n-2} \|\nabla u\|_2 = \frac{2}{n-2} \left( \int_D u(x) \Delta u(x) dx \right)^{\frac{1}{2}} \\ &\leq \frac{2}{n-2} \left( \int_D u^2(x) V(x) dx \right)^{\frac{1}{2}} \\ &\leq \frac{2}{n-2} \|u\|_{\frac{2n}{n-2}} \|V^{\frac{1}{2}}\|_n \\ &= \frac{2}{n-2} \|u\|_{\frac{2n}{n-2}} \|V\|_{\frac{n}{2}}^{\frac{1}{2}} \end{aligned}$$

In the step before the last we have applied Hölder's inequality with  $\frac{1}{2} = \frac{n-2}{2n} + \frac{1}{n}$ . To summarize:

$$\|u\|_{\frac{2n}{n-2}} \leq \frac{2}{n-2} \|u\|_{\frac{2n}{n-2}} \|V\|_{\frac{n}{2}}^{\frac{1}{2}}. \quad (4.8)$$

This implies iii). For ii), if  $V(x) \in L^r(D)$  for some  $r > \frac{n}{2}$ , then Hölder's inequality (with  $\frac{2}{n} = \frac{1}{r} + (\frac{2}{n} - \frac{1}{r})$ ), and (4.8) yield the following inequality, which implies ii).

$$\|u\|_{\frac{2n}{n-2}} \leq \frac{2}{n-2} \|u\|_{\frac{2n}{n-2}} |D|^{\frac{1}{n} - \frac{1}{2r}} \|V\|_r^{\frac{1}{2}}.$$

The authors originally proved Theorem 4.1 in a less direct manner, but obtained exactly the same lower bound for  $\|V\|_{\infty}$ . We do not know if it is sharp, but the following example shows it is not extremely far from that. If  $D = B_1(0)$ , then (4.5) implies  $\|V\|_{\infty} \geq 4$ . Let  $u_t(x) = r^2 e^{-tr^2}$ , with  $t = \frac{1}{2}(3 - \sqrt{5})$  and  $r = \sqrt{x^2 + y^2}$ . Then  $u_t(1) - u_t(r)$  vanishes on the unit circle.

In polar coordinates, its Laplacian is  $-\left(\partial_{rr}^2 u_t(r) + \frac{1}{r}\partial_r u_t(r)\right)$ , or

$$4e^{-r^2 t} \left( t(3 - r^2 t) r^2 - 1 \right).$$

We chose  $t = \frac{1}{2}(3 - \sqrt{5})$  so that this Laplacian vanishes for  $r = 1$ , and

$$V(r) = -\frac{\Delta u_t(r)}{u_t(1) - u_t(r)} = \frac{4e^t (t(3 - r^2 t) r^2 - 1)}{e^t r^2 - e^{r^2 t}}$$

is bounded. It is not too difficult to see that  $V$  is decreasing in  $[0, 1]$ . So, the maximum of  $V$  is  $V(0) = 4e^{\frac{1}{2}(3-\sqrt{5})} \sim 5.86$ .

The argument used to prove Part iii) of Theorem 4.1 can be generalized to prove the following unique continuation theorem.

**Proposition 4.2** *Let  $c = c(p, q, D)$  denotes the best constant in the Sobolev inequality*

$$\|u\|_{L^q(D)} \leq c(p, q, D) \|\Delta u\|_{L^p(D)} \tag{4.9}$$

where  $u \in W_0^{2,p}(D)$  and  $\frac{1}{q} \geq \frac{1}{p} - \frac{2}{n}$ . Then the differential inequality  $|\Delta u| \leq V(x)|u(x)|$  has only the trivial solution  $u \equiv 0$  if

$$c(p, q, D) \|V\|_{L^{\frac{n}{2}}(D)} < 1.$$

*Proof.* By (4.9) and Hölder's inequality,

$$\|u\|_q \leq c(p, q, D) \|\Delta u\|_p \leq c(p, q, D) \|V\|_{L^{\frac{n}{2}}} \|u\|_q$$

and if  $c(p, q, D) \|V\|_{L^{\frac{n}{2}}} < 1$ , then  $u \equiv 0$ .

## Section 5. Open problems and conjectures.

Our first conjecture is somewhat related to Theorem 4.1.i, and the isoperimetric inequality, about the possible area inside a curve. Recall that if  $\Delta u = 0$ , then  $Z$  cannot contain a closed loop. Can  $Z$  contain a C-shaped curve that is ‘almost’ closed?

**5.1. Isoperimetric Conjecture:** Suppose  $\Delta u = 0$  on  $\mathbf{C}$ , and that  $u > 0$  on  $B_1(0)$ . Suppose that  $L$  is a line segment in the plane of length  $\epsilon$  such that  $L \cup Z$  encloses  $B_1(0)$  (eg it does not intersect  $B_1(0)$ , but it disconnects the origin from infinity). Then  $\epsilon > c > 0$  for some absolute constant  $c$ .

Unfortunately, we have heuristic evidence that this is false, as stated. Consider the Poisson kernel  $P(z)$  for  $B_1(0)$ , which is positive on the open unit ball, and vanishes on the boundary, except at  $z = 1$ . It does not directly disprove the conjecture, since it is not harmonic at  $z = 1$ . But if  $u$  is a Taylor polynomial of  $P$  (roughly  $u = \operatorname{Re} 1 + z + z^2 + \dots z^n$ ), then  $u$  is harmonic on the plane. Computer experiments indicate that  $u$  has a level curve  $Z$  which almost encloses  $B_{1/4}(0)$ , with a small gap near  $z = 1$ , with length  $\epsilon \rightarrow 0$ , as  $n \rightarrow \infty$ . So, the conjecture seems to fail, but it still seems plausible for harmonic polynomials of fixed degree.

**5.2. Conjecture:** Suppose that in Theorem 3.5, the hypothesis  $\Delta v = 0$  is weakened to  $V \in L^\infty$  (see (1.1)). Is it still true that  $v$  must vanish on  $R^1$  ?

This conjecture seems plausible at first glance, since Theorem 3.5 is local in nature, and  $V \in L^\infty$  implies  $\Delta v = 0$  on  $Z$ . At this writing, the authors expect a negative answer; but our work is not quite conclusive. The next conjecture, which aims to extend Theorem 3.4 to higher dimensions, seems more likely.

**5.3. Conjecture:** Suppose that  $\Delta v = 0$  on  $R^n$ , and  $v = 0$  on  $B_1^{n-1}(0) \times \{0\} \subset R^n$ . Then  $v = 0$  on  $R^{n-1} \times \{0\}$ .

The thrust of this paper is that  $Z$  cannot wiggle arbitrarily. We have expressed this in terms of curvature and area, and would like some result about the local arc length of  $Z$ . But it seems very difficult to say much. Examples such as  $u = \operatorname{Re} z^k + \epsilon$  produce a set  $Z$  with very large total arc length inside the unit ball. Examples such as spirals show that curvature does not directly control local arc length.

But in the spirit of unique continuation, we conjecture that a large concentration of arc length cannot occur in only one small region of the plane.

**5.4. Arc Length Conjecture:** Suppose that  $u$  is harmonic on the unit ball  $B$ , and that  $Z$  is a finite union of rectifiable curves there, of total length  $|Z|$ . Then

$$|Z \cap B_{1/2}(0)| < 0.99|Z|.$$

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