Unique continuation for nonnegative solutions of Schrödinger type inequalities

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Abstract

We prove a sharp unique continuation theorem for nonnegative $H^{2,1}$ solutions of the differential inequality $|\Delta u(x)| \leq C|x-x_0|^{-2}|u(x)|$ which vanish of finite order at x_0.

In this paper we state and prove a unique continuation Theorem for the nonnegative $H^{2,1}_{loc}(\Omega)$ solutions of the differential inequality

$$|\Delta u(x)| \leq C|x - x_0|^{-2}|u(x)|, \qquad x, \ x_0 \in \Omega. \tag{1.1}$$

Here Ω is an open and connected domain of $\mathbf{R}^n$, with $n \geq 2$. Under suitable assumptions on the constant C in (1.1), we prove that every nonnegative solution of the differential inequality (1.1) which vanishes of order
$\omega >$max$\{n, 3\}$ at $x_0 \in \Omega$ is $\equiv 0$. This means for us that

$$\lim_{\epsilon\to 0} \epsilon^{-\omega}||u||_{L^1(B(x_0,\,\epsilon))} = 0, \tag{1.2}$$

where $B(x_0, \epsilon)$ is the ball centered at x_0 with radius ϵ.

It is not difficult to prove that if $u \in C^{m+1}(\Omega)$ and $D^\alpha u(x_0) = 0$ for every $|\alpha| \leq m$, then (1.2) holds whenever $\omega < m + n + 1$. Conversely, if $u \in C^m(\Omega)$ and (1.2) holds for $\omega \geq n + m$, then $D^\alpha u(x_0) = 0$ for every $|\alpha| \leq m$.

Recall that the differential inequality (1.1) has the *strong unique continuation property* if $u \equiv 0$ in Ω whenever u vanishes to infinite order at x_0, which means, for all $M > 0$,

$$\lim_{\epsilon\to 0} \epsilon^{-M}||u||_{L^1(B(x_0,\,\epsilon))} = 0. \tag{1.3}$$

The strong unique continuation property of the solutions of (1.1) and more general second order differential inequalities, has been investigated by many Authors. The literature on the subject is quite extensive, and we will not attempt to survey it. We just cite the survey paper of T. Wolff [W] and the references cited there.

To the best of our knowledge, most of the known results are in $H^{2,p}_{loc}(\Omega)$, with $p \geq \frac{2n}{n+2}$, and the unique continuation of $H^{2,1}_{loc}(\Omega)$ solutions of second order differential equations and inequalities has been discussed in very few papers. A notable example is the work of Garofalo and Lin [GL], (but see also [FGL] and [Z]).

In [GL], the Authors prove that the $H^{2,1}(\Omega)$ solutions of the equation $\operatorname{div}(A(x)\nabla u(x)) = 0$, ($A(x)$ a Lipschitz, positive definite matrix), and of the Schrodinger equation $-\Delta u(x) + C|x|^{-2}u(x) = 0$, where C is a constant, have the strong unique continuation property. The proofs are based on the "doubling condition" $\int_{B(x_0,2\epsilon)} u^2\,dx \leq c\int_{B(x_0,\epsilon)} u^2\,dx$, where c is an appropriate constant.

$H^{2,p_0}(\Omega)$, with $p_0 = \frac{2n}{n+2}$, is a remarkable space because the Sobolev embedding theorem maps $H^{2,p_0}_{loc}(\Omega)$ into $L^{p_0'}(\Omega)$, where $p_0' = \frac{2n}{n-2}$ is the dual exponent of p_0. This property of the exponents is crucial to prove Carleman-type inequalities.

A Carleman type inequality is an a priori weighted estimate of the form

$$||e^{\tau\psi(x)}u||_{L^p(\Omega)} \leq c||e^{\tau\psi(x)}\Delta u||_{L^q(\Omega)}, \qquad u \in C_0^\infty(\Omega), \tag{1.4}$$

where p and q satisfy the necessary condition[1] $\frac{2}{n} = \frac{1}{q} - \frac{1}{p}$, ψ is suitable "weight" function, τ is a real parameter that is allowed to go to $+\infty$, and c is a constant that does not depend on u and does not increase with τ.

Carleman inequalities have been a fundamental tool to prove unique continuation theorems for almost 70 years. In 1939 T. Carleman used for the first time an inequality of the form of (1.4) to show that the solutions of the differential inequality (1.1) have the strong unique continuation property in $H^{2,2}(\mathbf{R}^2)$, whenever $|V(x)| \in L^\infty_{loc}(\mathbf{R}^2)$, ([C]), and since then this original idea has permeated almost all developments in the subject.

[1]We can see that the condition on p and q is necessary by using a standard homogeneity argument.

Our Theorem can be proved without the aid of Carleman estimates. It can be stated as follows.

Theorem 1 *Let Ω be an open and connected set of $\mathbf{R}^n$, with $n \geq 2$. Let $u \in H^{2,1}_{loc}(\Omega)$ be a nonnegative solution of the differential inequality*

$$|\Delta u(x)| \leq C|x - x_0|^{-2}|u(x)|, \qquad x,\ x_0 \in \Omega. \tag{1.5}$$

Suppose that u vanishes at x_0 of order $\omega > \max\{n, 3\}$, (see (1.2)), and that

$$0 < C < C_\omega = (\omega - 2)(\omega - n). \tag{1.6}$$

Then, $u \equiv 0$ in Ω. C_ω cannot be replaced by any larger constant.

The assumption $u(x) \geq 0$ is crucial. Take, for example, $u(x) = \mathrm{Re}(z^N)$, with $z = x_1 + ix_2$. So, $u(x)$ vanishes of order N at the origin, but is harmonic.

The condition on V is not new. See e.g. [P], who proved the strong unique continuation of the $H^{2,2}_{loc}$ solution of (1.5). It is well known that the power of $|x - x_0|$ in (1.5) cannot be replaced by any larger power. See e.g. the survey paper [W] or counterexample of D. Jerison and K. Kenig in [JK].

The constant C_ω in (1.6) is sharp, for all admissible n and ω. Suppose $C'_\omega > C_\omega$. Let $u = |x|^m$, with $m > \omega - n > 0$. Then

$$||u||_{L^1(B(0,\,\epsilon))} = c\epsilon^{m+n}$$

so that (1.2) holds, and

$$\Delta u(x) = m(m + n - 2)|x|^{m-2} = C_{m+n}|x|^{-2}u(x)$$

We can make $m + n$ arbitrarily close to ω, so that $C_{m+n} < C'_\omega$, but clearly $u \not\equiv 0$.

Proof of Theorem 1. Without loss of generality, we can let $x_0 = 0$ and $\Omega = B_1 = B(0, 1)$. We want to prove that $\int_{B_1} u(x)\rho(x)\ dx = 0$, for some positive weight ρ, so that $u \equiv 0$ in B_1.

The main tool is Green's theorem. We will want w to vanish on the boundary of B_1, and $\triangle w > 0$ away from 0. There are inevitable problems at 0, which we will handle with cut-off functions and limits.

Assume that u vanishes to order ω as in (1.2). By continuity, there is an $m < \omega - 2$ such that $m > 1$, $m > n-2$ and $C < m(m-n+2)$. Let $r = |x|$ and let

$$w_m(x) = w_m(|x|) = r^{-m}(1-r)^m$$

The Laplacian of $w_m(x)$ can be easily computed, and is

$$\Delta w_m = m(1-r)^{m-2}\, r^{-2-m}(2+m-n+(n-3)r).$$

The Laplacian of $w_m(x)$ is positive in $(0,1]$. More specifically,

$$(m+n-2)(1-r)^{m-2}r^{-m-2} < \Delta w_m(x) < m(m-1)(1-r)^{m-2}\, r^{-2-m}. \tag{1.7}$$

Let $\chi(r) \in C^1(0, +\infty)$ be a cutoff function which is $\equiv 0$ in $[0,1]$ and is $\equiv 1$ in $[2, +\infty)$. For example, we can take $\chi(r) = 3(r-1)^2 - 2(r-1)^3$, for $1 \le r \le 2$.

Let $0 < \epsilon < 1$, and let $\chi_\epsilon(r) = \chi(\epsilon^{-1}r)$. Let $\tilde{u}(x) = u(x)\chi_\epsilon(x)$. Then,

$$\Delta\tilde{u}(x) = \Delta u(x)\chi_\epsilon(x) + u(x)\Delta\chi_\epsilon(x) + 2\nabla u(x)\nabla\chi_\epsilon(x).$$

We apply Green's theorem to the function $\Delta\tilde{u}(x)w_m(x)$. Since $w_m(x)$ and its radial derivative are zero on the boundary of B_1, and $\tilde{u}(x)$ and its radial derivative are $\equiv 0$ in B_ϵ, (recall that we have assumed that $\chi(1) = \chi'(1) = 0$), then the Gauss-Green theorem yields

$$\int_{B_1} \tilde{u}(x)\Delta w_m(x)dx = \int_{B_1} \chi_\epsilon(x)\Delta u(x) w_m(x)dx$$

$$+\int_{B_{2\epsilon}-B_\epsilon} \left(u(x)\Delta\chi_\epsilon(x) + 2\nabla u(x)\nabla\chi_\epsilon(x)\right) w_m(x)dx. \tag{1.8}$$

By (1.1),

$$\chi_\epsilon(x)\Delta u(x)w_m(x) \le \tilde{u}(x)V(x)w_m(x) \le C\tilde{u}(x)r^{-2}w_m(x).$$

By (1.8),

$$\int_{B_1} \tilde{u}(x)(\Delta w_m(x) - Cr^{-2}w_m(x))dx$$

$$\le \int_{B_{2\epsilon}} u(x)\Delta\chi_\epsilon(x)w_m(x)dx + \int_{B_{2\epsilon}} 2\nabla u(x)\nabla\chi_\epsilon(x)w_m(x)dx. \tag{1.9}$$

Let us summarize this inequality as $I_A \le I_B + I_C$ and first study I_A. The function

$$\rho(x) = \Delta w_m(x) - Cr^{-2} w_m(x)$$
$$= (1-r)^{m-2}\, r^{-2-m} \left\{ m(2+m-n+(n-3)r) - C(1-r)^2 \right\}$$

is positive in B_1. Indeed, the function $s(r) = m(2+m-n+(n-3)r) - C(1-r)^2$ attains its minimum either at $r=0$ or at $r=1$. Since $s(1) = m(m-1) > 0$, and $s(0) = m(m-n+2) - C > 0$, then $s(r) > K > 0$ on B_1. So,

$$I_A \ge K \int_{B_1} \tilde{u}(x)(1-r)^{m-2}\, r^{-2-m} dx. \tag{1.10}$$

Integrating by parts,

$$I_C = 2 \int_{B_{2\epsilon}-B_\epsilon} \nabla u(x) \nabla \chi_\epsilon(x) w_m(x) dx$$
$$= -2 \int_{B_{2\epsilon}} u(x) \Delta \chi_\epsilon(x) w_m(x) dx - 2 \int_{B_{2\epsilon}} u(x) \nabla \chi_\epsilon(x) \nabla w_m(x) dx.$$

Therefore,

$$I_B + I_C = - \int_{B_{2\epsilon}-B_\epsilon} u(x) \left(\Delta \chi_\epsilon(x) w_m(x) + 2 \nabla \chi_\epsilon(x) \nabla w_m(x) \right) dx.$$

We let c denote a sufficiently large constant independent of u and ϵ. Then $|\Delta \chi_\epsilon(r)| + \le c\epsilon^{-2}$, $|\nabla \chi_\epsilon(r)| \le c\epsilon^{-1}$, and $|w'_m(r)| = m\,(1-r)^{m-1}\ r^{-m-1} \le cm\epsilon^{-m-1}$ on $B_{2\epsilon} - B_\epsilon$, so

$$I_B + I_C \le c\epsilon^{-m-2} \int_{B_{2\epsilon}} u(x) \le c\epsilon^{-m+\omega-2} \tag{1.11}$$

for all small ϵ, by (1.2). So, combining (1.9), (1.10) and (1.11),

$$K \int_{B_1} \tilde{u}(x)(1-r)^{m-2} r^{-2-m} dx \le c\epsilon^{-m+\omega-2}. \tag{1.12}$$

Now let $\epsilon \to 0$. This shows that $\int_{B_1} u(x)(1-r)^{m-2} r^{-2-m} dx = 0$, which implies $u \equiv 0$ in B_1.

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