

Unique continuation for elliptic operators with non multiple characteristics

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Abstract

In this paper we prove a unique continuation theorem for elliptic operators of the form $P(D) + V(x)$, where $P(D)$ has order $m \geq 2$ and simple complex characteristics, and $V(x) \in L^{\frac{n}{m}}(\mathbf{R}^n)$. To prove our main theorem we use a restriction theorem for the Fourier transform to manifolds of codimension 2.

Introduction

Let $P(D)$ be an elliptic operator with real constant coefficients of order m , where $2 \leq m \leq n$ and $n \geq 3$, and let $P_m(D)$ be its principal part. In this paper we prove a "global" unique continuation theorem for a class of operators of the form $P(D) + V(x)$.

Theorem 0.1 *Suppose that $V \in L^{\frac{n}{m}}(\mathbf{R}^n)$. Suppose that $P(D)$ has simple complex characteristics in the direction ν , in the sense that the polynomial $\tau \rightarrow P_m(\zeta + i\tau\nu)$, $\tau \in \mathbf{C}$, has only simple zeroes for each fixed $\zeta \in \mathbf{R}^n / \{0\}$. Then, every $u \in H^{m,p}(\mathbf{R}^n)$,*

$$\frac{1}{2} \leq \frac{1}{p} \leq \frac{m}{n} + \frac{n-2}{2(n+2)}, \quad (0.1)$$

which is supported on one side of the hyperplane $\{x : \langle x, \nu \rangle = 0\}$, and solves

$$|P(D)u(x)| \leq |V(x)u(x)|, \quad (0.2)$$

is $\equiv 0$ in \mathbf{R}^n .

By $H^{m,p}(\mathbf{R}^n)$ we mean the space of functions with m derivatives in $L^p(\mathbf{R}^n)$.

It is well known that the multiplicity of the complex characteristics of $P(D)$ can affect the unique continuation properties of the solutions of (0.2). The counterexamples of A. Plis, (see [P]), show that an elliptic operator whose principal part has constant coefficient and sixfolds complex characteristics of constant multiplicity can have solution with compact support, and since then a great effort has been devoted to establish the unique continuation properties of elliptic operator of order > 2 with multiple characteristics. When $V \in L^\infty(\mathbf{R}^n)$ and the complex characteristics of $P(D)$ are at most double, the unique continuation properties of the $H_{loc}^m(\mathbf{R}^n)$ solutions of (0.2) are more or less well understood since many years, ([Ca], [H1]). The unique continuation properties of an elliptic operators with complex characteristics of multiplicity ≥ 2 are more difficult to understand, (see [O] and the bibliography there).

A distinctive feature of our problem is the fact that the "potential" $V(x)$ is not bounded. In this case, very little seems to be in the literature, ([DC], [Wa], [CG]). In [DC] the author proved that the differential inequality (0.2) does not have solutions with compact support in $H^{m,p}(\mathbf{R}^n)$ provided that the characteristics of $P(D)$ are simple and the symbol of P_m satisfies an additional convexity assumption. The Theorem that we prove in this paper is more general, although far from being optimal. For example, our techniques do not allow us to prove that the conclusions of Theorem 0.1 also hold when the complex characteristics of $P_m(D)$ are at most double. Moreover we do not know whether our Theorems fails or not when $V \in L^s(\mathbf{R}^n)$, $1 \leq s < \frac{n}{m}$. However, the issue is very difficult, and has not been yet addressed even for the Laplace operator, (see [W]).

We shall prove Theorem 0.1 using the method of Carleman inequalities. Specifically, Carleman's method says roughly that if an *a priori* inequality of the form

$$\|e^{\tau\phi(x)}f\|_{L^q(\mathbf{R}^n)} \leq C\|e^{\tau\phi(x)}P(D)f\|_{L^p(\mathbf{R}^n)}, \quad f \in C_0^\infty(\mathbf{R}^n) \quad (0.3)$$

for a suitable weight ϕ and $q > p$, is valid for a constant C independent of τ which is allowed to tend to infinity, then $P(D) - V$ has the unique continuation property provided that $V \in L^r(\mathbf{R}^n)$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$.

There is a large literature concerning Carleman estimates and their application to uniqueness questions.

An important tool of our proof is a sharp restriction theorem for the Fourier transform to a class of manifolds of codimension 2.

Let M denote a submanifold of \mathbf{R}^n equipped with a compactly supported measure $d\sigma$. Let \mathcal{R} denote a restriction operator

$$\mathcal{R}f(\eta) = \int_{\mathbf{R}^n} e^{-i\langle x, \eta \rangle} f(x) dx, \quad \eta \in M, \quad f \in \mathcal{S}(\mathbf{R}^n). \quad (0.4)$$

We say that $(M, d\sigma)$ has the $L^p \rightarrow L^q$ restriction property if

$$\|\mathcal{R}f\|_{L^q(M, d\sigma)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}. \quad (0.5)$$

The connection between the restriction properties of hypersurfaces and Carleman inequalities seems to be first observed by Hormander [H1]. Later, C. E. Kenig, A. Ruiz, and C. D. Sogge [KRS], used the restriction properties of nonsingular real quadratic forms in \mathbf{R}^n to prove uniform Sobolev inequalities for second order operators with constant coefficients, which immediately implies (0.3) when ϕ is linear. However, it seems that the restriction properties of manifold of codimension 2 have been used in connection to Carleman type estimate for the first time by the author [DC].

The paper is organized as follows: in Section 1 we state the preliminaries and prove some technical Lemma. In Section 2 we prove Theorem 0.1.

In what follows we shall use the convention that χ denotes a smooth cutoff function which is not necessarily the same at each occurrence. Also, we will denote by C a constant which may change from line to line. Finally, we would like to thank E. Sawyer, who provided financial support through his NSF grant while this paper was in preparation.

1 Preliminaries

Let M denote a submanifold of \mathbf{R}^n equipped with a compactly supported measure $d\sigma$. Let \mathcal{R} denote the restriction operator (0.4). We say that $(M, d\sigma)$ has the $L^p \rightarrow L^2$ restriction property if (0.5) holds with $q = 2$. The case $q = 2$

is a special one, since one can check, using an observation of Stein and Tomas, (see [T]), that (0.5) is equivalent to

$$\|\widehat{d\sigma} * f\|_{L^{p'}(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}, \quad (1.1)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The inequality (1.1) is easier to handle than (0.5); for example, one can embed $f \rightarrow \widehat{d\sigma} * f$ into an analytic family of operator and obtain (1.1) by (complex) interpolation from a $L^2 \rightarrow L^2$ estimate and a $L^1 \rightarrow L^\infty$ estimate. For a description of the complex interpolation method see e.g. [SW].

While the $L^p \rightarrow L^2$ restriction properties of the Fourier transform to manifolds of codimension 1, and in particular the relationship between the Gaussian curvature of the manifold and the best exponent p_1 for which the $L^{p_1} \rightarrow L^2$ restriction theorem holds is well understood since many years, (see e.g. [S]), the restriction properties of the Fourier transform to manifolds of codimension ≥ 2 are still not well understood. A generalization of a homogeneity argument due to Knapp, (see e.g [C] or [DCI]), shows that if a $L^p \rightarrow L^2$ restriction theorem holds for a manifold of codimension 2, then $p \leq p_2 = \frac{2(n+2)}{n+6}$.

We shall state a necessary and sufficient condition for the the $L^{p_2} \rightarrow L^2$ restriction theorem. In what follows we will often denote $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_1) \in \mathbf{R}^n$ by (ζ_1, ζ') , $\zeta' \in \mathbf{R}^{n-1}$, $\zeta_1 \in \mathbf{R}$, or by $(\zeta_1, \zeta_2, \zeta'')$, $\zeta'' \in \mathbf{R}^{n-2}$, $\zeta_i \in \mathbf{R}$, unless otherwise specified.

Let $\zeta_0 \in M$. After perhaps a change of coordinates, M can be locally written as the graph of $\psi(\zeta'') = (\psi_1(\zeta''), \psi_2(\zeta''))$, where ψ_1 and ψ_2 are smooth in a neighborhood of ζ_0 . Let $H(\zeta, \theta) = \det(\cos \theta D^2(\psi_1)(\zeta) + \sin \theta D^2(\psi_2)(\zeta))$, where $D^2(\psi_j)$ denotes the Hessian matrix of ψ_j and $\theta \in [0, 2\pi)$. We state the following Theorem.

Theorem 1.1 *M has the $L^{p_2} \rightarrow L^2$ restriction property if and only if, for all $\zeta_0 \in M$ and all $\theta_0 \in [0, 2\pi)$, there exists $k \leq \frac{n}{2}$ such that*

$$\frac{d^k}{d\theta^k} H(\zeta_0, \theta_0) \neq 0, \quad (1.2)$$

or equivalently, if and only if

$$\int_{S^{n-1}} |H(\zeta_0, \theta)|^{-k} d\theta < \infty \quad \text{for some } k < \frac{n-2}{2} \quad (1.3)$$

In his thesis, M. Christ proved that (1.2) is necessary for the $L^{p_2} \rightarrow L^2$ restriction theorem. Fourteen years later the "if" part of Theorem 1.1 has been proved by G. Mockenhaupt in [M].

The manifolds that we shall consider in this paper are of the following form: let $\lambda(\zeta'), \mu(\zeta') \in C^\infty(\mathbf{R}^{n-1}/0)$ denote two homogeneous functions of degree one. Suppose that $\mu(\omega)$ never vanishes on S^{n-2} . Let

$$S = \{\zeta \in \mathbf{R}^n : \zeta_1 = \lambda(\zeta'), 1 = \mu(\zeta')\}. \quad (1.4)$$

S is a submanifold of \mathbf{R}^n of codimension 2 that can be viewed as the intersection of the "cone" $\{\zeta : \zeta_1 = \lambda(\zeta')\}$ and the "cylinder" $\{\zeta : 1 = \mu(\zeta')\}$.

In what follows we would find preferable to write S as the graph of smooth functions. Let $\zeta_0 \in S$. Note that $\zeta'_0 \neq (0, 0, \dots, 0)$, since $\mu(\zeta'_0) = 1$.

By the Euler homogeneity relation, the gradient of $\mu(\zeta')$ does not vanish at ζ'_0 . After perhaps a change of coordinates, we can assume that $\nabla\mu(\zeta'_0) = (1, 0, \dots, 0)$; we can also assume, without loss of generality, that either $\nabla\lambda(\zeta'_0) = (0, \dots, 0, 0)$, or $\nabla\lambda(\zeta'_0) = (0, \dots, 0, 1)$. By the implicit function theorem there exists $h_2(\zeta'')$ such that

- i) $h_2(\zeta'')$ is smooth in a neighborhood of ζ''_0
- ii) $\zeta'_0 = (h_2(\zeta''_0), \zeta''_0)$,
- iii) $\mu(h_2(\zeta''), \zeta'') \equiv 1$.

Let $h_1(\zeta'') = \lambda(h_2(\zeta''), \zeta'')$. Then S can be locally viewed as the graph of $h(\zeta'') = (h_1(\zeta''), h_2(\zeta''))$. We prove the following technical Lemma

Lemma 1.2 *Let S be defined as in (1.4). Let $\zeta_0 \in M$. Let $h_2(\zeta'')$ be as in i), ii) and iii) above. Let $h_1(\zeta'') = \lambda(h_2(\zeta''), \zeta'')$. Then,*

$$\zeta \rightarrow \frac{\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))}{\zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))}$$

is bounded in a neighborhood of ζ_0 .

Proof. Let $G(t) = \zeta_1 - \lambda(th_2(\zeta'') + (1-t)\zeta_2, \zeta'') + i(1 - \mu(th_2(\zeta'') + (1-t)\zeta_2, \zeta''))$, $t \in \mathbf{R}$. G is a smooth function of t when ζ is in a neighborhood of ζ_0 that

does not contain the origin and $t \in (0, 1)$, and $G(0) = \zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))$. By Taylor's formula, $G(0) = G(1) + G'(1) + \frac{1}{2}G''(\bar{t})$ for some $\bar{t} \in (0, 1)$; thus,

$$\begin{aligned} \zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta')) &= \zeta_1 - h_1(\zeta'') \\ &- \left(\frac{\partial}{\partial \zeta_2} \lambda(h_2(\zeta''), \zeta'') + i \left(\frac{\partial}{\partial \zeta_2} \mu(h_2(\zeta''), \zeta'') \right) \right) (h_2(\zeta'') - \zeta_2) \\ &\quad + \frac{1}{2} (h_2(\zeta'') - \zeta_2)^2 R(\zeta, \bar{t}), \end{aligned}$$

where $R(\zeta, \bar{t}) = -\frac{\partial^2}{\partial \zeta_2^2} (\lambda + i\mu)(\bar{t}h_2(\zeta'') + (1 - \bar{t})\zeta_2, \zeta'')$. We recall that $\frac{\partial}{\partial \zeta_2} \mu(\zeta_0') = 1$ and $\frac{\partial}{\partial \zeta_2} \lambda(\zeta_0') = 0$. Thus, if we take $\epsilon > 0$ sufficiently small, we can take $\delta > 0$ so that, if $|\zeta'' - \zeta_0''| < \delta$, then $\left| \frac{\partial}{\partial \zeta_2} \lambda(h_2(\zeta''), \zeta'') \right| \leq \epsilon$ and $\frac{\partial}{\partial \zeta_2} \mu(h_2(\zeta''), \zeta'') \geq 1 - \epsilon$. If δ is sufficiently small, we can also assume that $\frac{1}{2}(h_2(\zeta'') - \zeta_2)^2 |R(\zeta, \bar{t})| \leq \epsilon |h_2(\zeta'') - \zeta_2|$ when $|\zeta' - \zeta_0'| < \delta$ and $\bar{t} \in (0, 1)$. Thus,

$$\begin{aligned} &|\zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))| \\ &\geq \left| \zeta_1 - h_1(\zeta'') - i \left(\frac{\partial}{\partial \zeta_2} \mu(h_2(\zeta''), \zeta'') \right) (h_2(\zeta'') - \zeta_2) \right| - \epsilon |h_2(\zeta'') - \zeta_2| \\ &\geq (1 - 2\epsilon) (|\zeta_1 - h_1(\zeta'')| + |h_2(\zeta'') - \zeta_2|) \geq (1 - 2\epsilon) |\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|. \end{aligned}$$

From the above immediately follows that

$$\left| \frac{\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))}{\zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))} \right| \leq \frac{1}{1 - 2\epsilon},$$

when ζ is in a neighborhood of ζ_0 , as required.

2 Proof of the main Theorems

In this section we will prove Theorem 0.1 as a consequence of the following Carleman type inequality:

Theorem 2.1 *Let p, P and V be defined as in Theorem 0.1. Then, for every $\tau > 0$ and for every $u \in C_0^\infty(\mathbf{R}^n)$, one has*

$$\|e^{\tau \langle \nu, x \rangle} u\|_{L^q(\mathbf{R}^n)} \leq C \|e^{\tau \langle \nu, x \rangle} P(D)u\|_{L^p(\mathbf{R}^n)}, \quad (2.1)$$

where

$$\frac{1}{p} - \frac{1}{q} = \frac{m}{n}, \quad (2.2)$$

$$\frac{1}{2} \leq \frac{1}{p} \leq \frac{m}{n} + \frac{n-2}{2(n+2)} \quad (2.3)$$

Remark: A standard homogeneity argument show that (2.2) is necessary for (2.1). This is the reason why we must take $V \in L^{\frac{n}{m}}(\mathbf{R}^n)$ in (0.2).

Let us recall why Theorem 2.1 implies Theorem 0.1. Let U be a solution of (0.2). For simplicity of notation we shall assume that U is supported in the half space $\{x = (x', x_1) : x_1 > 0\}$. We use the following special case of (2.1):

$$\|e^{-\tau x_1} u\|_{L^q(\mathbf{R}^n)} \leq C \|e^{-\tau x_1} P(D)u\|_{L^p(\mathbf{R}^n)}. \quad (2.4)$$

Since $\tau x_1 \geq 0$ on the support of U , and since $C_0^\infty(\mathbf{R}^n)$ is dense in $H^{m,p}(\mathbf{R}^n)$, it is easy to see that U satisfies (2.4).

To prove that $U \equiv 0$ it is sufficient to prove that there is a $\rho > 0$ so that $U \equiv 0$ in the strip $S_\rho = \{x \in \mathbf{R}^n : x_1 \leq \rho\}$. Take $\rho > 0$ so small that, if V is as above and C is as in (2.4),

$$C \|V\|_{L^{\frac{n}{m}}(S_\rho)} \leq \frac{1}{2}. \quad (2.5)$$

If one uses Holder's inequality, along with (2.1) and (2.5), and the fact that $|P(D)U| \leq |VU|$, then one has the following string of inequalities for every $\tau > 0$.

$$\begin{aligned} \|e^{-\tau x_1} U\|_{L^q(S_\rho)} &\leq C \|e^{-\tau x_1} P(D)U\|_{L^p(\mathbf{R}^n)} \\ &\leq C \|e^{-\tau x_1} VU\|_{L^p(S_\rho)} + C \|e^{-\tau x_1} P(D)U\|_{L^p(\mathbf{R}^n/S_\rho)} \\ &\leq \frac{1}{2} \|e^{-\tau x_1} U\|_{L^{p'}(S_\rho)} + C \|e^{-\tau x_1} P(D)U\|_{L^p(\mathbf{R}^n/S_\rho)}. \end{aligned}$$

Hence,

$$\|e^{-\tau x_1} U\|_{L^q(S_\rho)} \leq 2C \|e^{-\tau x_1} P(D)U\|_{L^p(\mathbf{R}^n/S_\rho)},$$

and consequently,

$$\|e^{\tau(\rho-x_1)} U\|_{L^q(S_\rho)} \leq 2C \|P(D)U\|_{L^p(\mathbf{R}^n)}.$$

Since the above inequality holds for every $\tau > 0$, this forces $U \equiv 0$ in S_ρ .

The rest of this section will be devoted to the proof of Theorem 2.1. After replacing u by $e^{(x,\nu)}v$, (2.1) is equivalent to

$$\|v\|_{L^q(\mathbf{R}^n)} \leq C\|P(D + i\tau\nu)u\|_{L^p(\mathbf{R}^n)}, \quad (2.6)$$

and since the inverse Fourier transform of $(P(\zeta + i\tau\nu))^{-1}$ is a fundamental solution for the operator $P(D + i\tau\nu)$, it follows that the inequality (2.6) would be a consequence of

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle \zeta, x \rangle}}{P(\zeta + i\tau\nu)} \hat{f}(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}. \quad (2.7)$$

Let $P(\zeta)$ denote the symbol of P . We use the same reductions as in [DC1]. A standard scaling argument reduces (2.7) to

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle \zeta, x \rangle}}{\tau^{-m}P(\tau\zeta + i\tau\nu)} \hat{f}(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}. \quad (2.8)$$

Note that $\tau^{-m}P(\tau\zeta + i\tau\nu) = P_m(\zeta + i\nu) + \mathcal{O}(\tau^{-1})$ when $|\zeta|$ is bounded. Thus, if τ is sufficiently large, the polynomial $\zeta \rightarrow \tau^{-m}P(\tau\zeta + i\tau\nu)$ vanishes in a small neighborhood of the (compact) set where $\zeta \rightarrow P_m(\zeta + i\nu)$ vanishes.

With these observations in mind, we will see that it is immaterial to consider $P_m(\zeta + i\nu)$ rather than $\tau^{-m}P(\tau\zeta + i\tau\nu)$ in what follows, and thus, we assume that $P(D) = P_m(D)$ for the sake of simplicity.

Without loss of generality, let $\nu = (1, 0, \dots, 0)$. After perhaps a rotation of coordinates,

$$P(\zeta) = \zeta_1^m + \sum_{j=0}^{m-1} \zeta_1^j Q_j(\zeta'),$$

where the Q_j 's are real homogeneous polynomials of degree $m - j$. The roots of $\zeta_1 \rightarrow P(\zeta)$, $\lambda_1(\zeta') + i\mu_1(\zeta'), \dots, \lambda_m(\zeta') + i\mu_m(\zeta')$, are smooth and homogeneous of degree 1 in $\mathbf{R}^{n-1}/\{0\}$. Since $P(D)$ is elliptic with real coefficients, the μ_j 's vanish only at the origin, and $\lambda_j(\zeta') + i\mu_j(\zeta')$ and $\lambda_j(\zeta') - i\mu_j(\zeta')$ are both roots of $\zeta_1 \rightarrow P(\zeta_1, \zeta')$. Then,

$$P(\zeta) = \prod_{j=1}^{\frac{m}{2}} (\zeta_1 - \lambda_j(\zeta') \pm i\mu_j(\zeta')),$$

where $\mu_j(\zeta') > 0$, $\zeta' \in \mathbf{R}^{n-1}/\{0\}$, and consequently

$$P(\zeta + i\nu) = \prod_{j=1}^{\frac{m}{2}} (\zeta_1 - \lambda_j(\zeta') + i(1 \pm i\mu_j(\zeta'))). \quad (2.9)$$

The factors of the form $\zeta_1 - \lambda_j(\zeta') + i(1 + \mu_j(\zeta'))$ are the "good" factors which never vanish, but the "bad" factors $\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))$ vanish on the (compact) manifolds

$$S_j = \{\zeta : \zeta_1 = \lambda_j(\zeta'), 1 = \mu_j(\zeta')\}, \quad j \leq \frac{m}{2}.$$

Since $P(D)$ has simple complex characteristics in the direction ν , then $S_j \cap S_k = \emptyset$ when $j \neq k$, and we can choose cutoff functions $\chi_j(\zeta) \in C_0^\infty(\mathbf{R}^n)$ which are $\equiv 1$ in a neighborhood of S_j and $\equiv 0$ in a neighborhood of S_k when $k \neq j$.

Let $\chi_0(\zeta) = 1 - (\chi_1(\zeta) + \cdots + \chi_{\frac{m}{2}}(\zeta))$. Thus,

$$\begin{aligned} & \left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta)}{P(\zeta + i\nu)} d\zeta \right\|_{L^q(\mathbf{R}^n)} \\ & \leq \left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta)}{P(\zeta + i\nu)} \chi_0(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} + \sum_{j=1}^{\frac{m}{2}} \left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta)}{P(\zeta + i\nu)} \chi_j(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)}. \end{aligned} \quad (2.10)$$

We shall prove the following Lemmas

Lemma 2.2 *The following inequality holds for $p < \frac{n}{m}$ and $q \geq p$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{m}{n}$, and for all $f \in C_0^\infty(\mathbf{R}^n)$.*

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta)}{P(\zeta + i\nu)} \chi_0(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}. \quad (2.11)$$

Lemma 2.3 *The following inequality holds for $q \geq \frac{2(n+2)}{n-2}$ and $p \leq 2$, and for all $f \in C_0^\infty(\mathbf{R}^n)$.*

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta)}{P(\zeta + i\nu)} \chi_j(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}, \quad j \leq \frac{m}{2}. \quad (2.12)$$

(2.10), and hence (2.1), immediately follow from Lemma 2.2 and Lemma 2.3.

Proof of Lemma 2.2. Since $P(\zeta + i\nu)$ does not vanish on the support of χ_0 , then $\frac{\chi_0(\zeta)}{P(\zeta + i\nu)}$ is the symbol of a pseudodifferential operator of order $-m$, which can be extended to a bounded operator from $L^p(\mathbf{R}^n)$ to $H^{m,p}(\mathbf{R}^n)$. By the Sobolev embedding theorem,

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \widehat{f}(\zeta)}{P(\zeta + i\nu)} \chi_0(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)},$$

where $\frac{1}{p} - \frac{1}{q} \leq \frac{m}{n}$, as required.

Proof of Lemma 2.3. By Hörmander's multiplier theorem, $\zeta \rightarrow \frac{\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))}{P(\zeta + i\nu)} \chi_j(\zeta)$ is a L^q Fourier multiplier for every $q > 1$ and all $j \leq \frac{m}{2}$, (see e.g. [So]). Thus, in order to prove (2.12), it is sufficient to prove (2.13) below.

$$\left\| \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \widehat{f}(\zeta)}{\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))} \chi_j(\zeta) d\zeta \right\|_q \leq C \|f\|_p, \quad j \leq \frac{m}{2}. \quad (2.13)$$

For simplicity we will omit the subscript j from now on. Let

$$T(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle}}{\zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))} \widehat{f}(\zeta) \chi(\zeta) d\zeta,$$

and let $S = \{\zeta \in \mathbf{R}^n : \zeta_1 = \lambda(\zeta'), \mu(\zeta') = 1\}$. Let $\zeta_0 \in S$. Since S is compact, there is no loss of generality if we assume that the support of χ is concentrated in a small neighborhood of ζ_0 . Let $h(\zeta'') = (h_1(\zeta''), h_2(\zeta''))$ be as in Lemma 1.2. Thus,

$$T(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle}}{\zeta_1 - h_1(\zeta'') + i(\zeta_1 - h_1(\zeta''))} m(\zeta) \widehat{f}(\zeta) \chi(\zeta) d\zeta, \quad (2.14)$$

where we have let

$$m(\zeta) = \frac{\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))}{\zeta_1 - \lambda(\zeta') + i(1 - \mu(\zeta'))}.$$

By Lemma 1.2, $\chi(\zeta)m(\zeta)$ is bounded if the support of χ is sufficiently small.

Let $0 < \sigma < 1$; consider the operators

$$L(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta) \chi(\zeta)}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^{1-\sigma}} d\zeta, \quad (2.15)$$

$$G(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle}}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^\sigma} M(\zeta) f(\zeta) \chi(\zeta) d\zeta, \quad (2.16)$$

where we have let $M(\zeta) = m(\zeta)e^{-i\text{Arg}(\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta'')))}$. Since $T = L \circ G$, if we prove that $\|G(f)\|_2 \leq C\|f\|_p$, and $\|L(f)\|_q \leq C\|f\|_2$, $q' \leq \frac{2(n+2)}{n+6}$, then the proof of (2.13) follows.

We consider G first. We recall that, by Lemma 1.2, $M(\zeta)$ is bounded. By Plancherel theorem,

$$\|G(f)\|_2^2 \leq C \int_{\mathbf{R}^n} \frac{|\hat{f}(\zeta)|^2}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^{2\sigma}} \chi(\zeta) d\zeta.$$

If we let $\frac{1}{p'} = 1 - \frac{1}{p}$, then $p' > 2$, and hence $\frac{1}{2} - \frac{1}{p'} > 0$. By Holder's inequality with exponents $\frac{1}{p'} + \frac{1}{s} = \frac{1}{2}$,

$$\|G(f)\|_2 \leq \|\hat{f}\|_{p'} \left(\int_{\mathbf{R}^n} \frac{\chi(\zeta)}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^{s\sigma}} d\zeta \right)^{\frac{1}{s}}, \quad (2.17)$$

The integral in (2.17) is finite if $\sigma s < 2$. By Hausdorff-Young inequality, $\|G(f)\|_2 \leq C\|f\|_p$ for every $p < 2$.

We shall now prove that L is bounded. Since the adjoint of L is \bar{L} , we prove that $\|L(f)\|_2 \leq C\|f\|_{q'}$, where $\frac{1}{q'} = 1 - \frac{1}{q}$.

Assume that h satisfies (1.2) of Theorem 1.1 when $\zeta \in \text{supp}(\chi)$. By Plancherel theorem,

$$\|L(f)\|_2^2 = \int_{\mathbf{R}^n} \frac{|\hat{f}(\zeta)|^2}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^{2(1-\sigma)}} \chi(\zeta) d\zeta.$$

If we let $\zeta_1 = s + h_1(\zeta'')$, $\zeta_2 = t + h_2(\zeta'')$,

$$\|L(f)\|_2^2 = \int_{\mathbf{R}^n} \frac{|\hat{f}(s + h_1(\zeta''), t + h_2(\zeta''), \zeta'')|^2}{|s + it|^{2(1-\sigma)}} \chi(s, t, \zeta'') ds dt d\zeta''.$$

Without loss of generality $\chi(s, t, \zeta'')$ can be written as $\chi(r)\chi(s)\chi(|\zeta'' - \zeta_0''|)$, where the support of the new χ is concentrated in a neighborhood of the origin. Thus,

$$\|L(f)\|_2^2 = \int_{\mathbf{R}^2} \frac{\chi(s)\chi(t)}{|s+it|^{2(1-\sigma)}} \int_{\mathbf{R}^{n-2}} |\widehat{\Lambda_{t,s}(f)}(h_1(\zeta''), h_2(\zeta''), \zeta'')|^2 \chi(|\zeta'' - \zeta_0''|) d\zeta'',$$

where we have set $\Lambda_{t,s}(f)(x) = e^{-i(tx_1+sx_2)} f(x)$. By assumption, S has the $L^{q'} \rightarrow L^2$ restriction property with $q' \leq p_0 = \frac{2(n+2)}{n+6}$. Note that $q \geq \frac{2(n+2)}{n-2}$. Thus,

$$\|L(f)\|_2^2 \leq C \|\Lambda_{t,s}(f)\|_{q'}^2 \int_{\mathbf{R}^2} \frac{\chi(s)\chi(t)}{|s+it|^{2(1-\sigma)}} ds dt \leq C \|f\|_{q'}^2,$$

since $|\Lambda_{t,s}(f)(x)| = |f(x)|$, and the above integral is finite for all $0 < \sigma < 1$.

When S does not have the $L^{p_0} \rightarrow L^2$ restriction property we use a little trick. Let $M = \{\zeta : \zeta_1 = \phi_1(\zeta''), \zeta_2 = \phi_2(\zeta'')\}$ be a smooth manifold of codimension 2. Suppose that $\zeta_0 \in M$ and that $\phi = (\phi_1, \phi_2)$ satisfies (1.2) of Theorem 1.1 when $\zeta \in \text{supp}(\chi)$. We let

$\zeta_1 \rightarrow \zeta_1 + h_1(\zeta'') - \phi_1(\zeta'')$, $\zeta_2 \rightarrow \zeta_2 + h_2(\zeta'') - \phi_2(\zeta'')$ in (2.14). Thus,

$$T(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle}}{\zeta_1 - \phi_1(\zeta'') + i(\zeta_2 - \phi_2(\zeta''))} M(\zeta, x_1, x_2) Hf(\zeta) \chi(\zeta) d\zeta, \quad (2.18)$$

where we have let $Hf(\zeta) = \hat{f}(\zeta_1 + h_1(\zeta'') - \phi_1(\zeta''), \zeta_2 + h_2(\zeta'') - \phi_2(\zeta''), \zeta'')$, and

$$\begin{aligned} M(\zeta, x_1, x_2) &= e^{i(x_1(h_1(\zeta'') - \phi_1(\zeta'')) + x_2(h_2(\zeta'') - \phi_2(\zeta'')))} \\ &\times m(\zeta_1 + h_1(\zeta'') - \phi_1(\zeta''), \zeta_1 + h_1(\zeta'') - \phi_1(\zeta''), \zeta''). \end{aligned}$$

Since $h(\zeta_0'') = \phi(\zeta_0'')$, the application $\zeta \rightarrow (\zeta_1 + h_1(\zeta'') - \phi_1(\zeta''), \zeta_2 + h_2(\zeta'') - \phi_2(\zeta''), \zeta'')$ maps a neighborhood of ζ_0 onto a neighborhood of ζ_0 , and $|M(\zeta, x_1, x_2)|$ is uniformly bounded, with bounds independent of x_1 and x_2 , when the support of χ is sufficiently small. Let $0 < \sigma < 1$; consider the operators

$$L(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle} \hat{f}(\zeta) \chi(\zeta)}{|\zeta_1 - \phi_1(\zeta'') + i(\zeta_2 - \phi_2(\zeta''))|^{1-\sigma}} d\zeta,$$

$$G(f) = \int_{\mathbf{R}^n} \frac{e^{i\langle x, \zeta \rangle}}{|\zeta_1 - \phi_1(\zeta'') + i(\zeta_2 - \phi_2(\zeta''))|^\sigma} \tilde{M}(x_1, x_2, \zeta) Hf(\zeta) \chi(\zeta) d\zeta,$$

where we have let $\tilde{M}(x_1, x_2, \zeta) = M(x_1, x_2, \zeta)e^{-i\text{Arg}(\zeta_1 - \phi_1(\zeta'') + i(\zeta_2 - \phi_2(\zeta''))}$. Since $T = L \circ G$, if we prove that $\|G(f)\|_2 \leq C\|f\|_p$, and $\|L(f)\|_q \leq C\|f\|_2$, $q' \leq \frac{2(n+2)}{n+6}$, then the proof of (2.12) follows.

L can be estimated as for the previous case. To estimate G , observe that by Plancherel theorem,

$$\|G(f)\|_2^2 \leq C \int_{\mathbf{R}^n} \frac{|Hf(\zeta)|^2}{|\zeta_1 - \phi_1(\zeta'') + i(\zeta_2 - \phi_2(\zeta''))|^{2\sigma}} \chi(\zeta) d\zeta. \quad (2.19)$$

We let $\zeta_1 \rightarrow \zeta_1 - h_1(\zeta'') + \phi_1(\zeta'')$, $\zeta_2 \rightarrow \zeta_2 - h_2(\zeta'') + \phi_2(\zeta'')$ in (2.16). We obtain

$$\|G(f)\|_2^2 \leq C \int_{\mathbf{R}^n} \frac{|\hat{f}(\zeta)|^2}{|\zeta_1 - h_1(\zeta'') + i(\zeta_2 - h_2(\zeta''))|^{2\sigma}} \chi(\zeta) d\zeta,$$

and after that $\|G(f)\|_2^2$ can be estimated as for the previous case. The proof of Lemma 2.3 is thus complete.

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