On the $L^p - L^q$ norm of the Hankel transform and related operators

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Abstract

We investigate the $L^p(\mathbf{R}^+) - L^q(\mathbf{R}^+)$ mapping properties of the operator

$$L^{\alpha}_{\nu,\,\mu} f(y) = y^{\mu} \int_0^{\infty} (xy)^{\nu} f(x) J_{\alpha}(xy) dx, \qquad f \in C_0^{\infty}(0,\,+\infty),$$

for suitable values of the parameters, and we evaluate the operator norm of $L^{\alpha}_{\nu,\mu}$ in some special and significant cases.

1 Introduction

We consider the following class of operators,

$$\mathcal{L} = \left\{ L^{\alpha}_{\nu,\mu} f(y) = y^{\mu} \int_0^{\infty} (xy)^{\nu} f(x) J_{\alpha}(xy) dx, \quad f \in C_0^{\infty}(0,\,+\infty) \right\}, \tag{1.1}$$

where $J_\alpha(r)$ denotes the usual Bessel function of the first kind, $\alpha \geq -\frac{1}{2}$ and ν and μ are real parameters.

In this paper we investigate the $L^p(\mathbf{R}^+) - L^q(\mathbf{R}^+)$ mapping properties of these operators for exponents $1 \leq p,\, q \leq \infty$.

We will consider real-valued L^p spaces, since a general theorem of J. Marcinkiewicz and A. Zygmund about vector-valued linear operators, (see [MZ]), implies that a bounded linear operator that maps a real-valued L^p space into a real valued L^q space also maps the complex valued version of the same spaces into themselves with the same norm.

The main results of this paper are the following:

Theorem 1.1 *$L^{\alpha}_{\nu,\,\mu}$ is bounded from $L^p(\mathbf{R}^+)$ to $L^q(\mathbf{R}^+)$ whenever $\alpha \geq -\dfrac{1}{2}$, and*

a) $1 \leq p \leq q \leq p'$, *and if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \quad \textit{and} \quad -\alpha - \frac{1}{p'} < \nu \leq \frac{1}{2}, \tag{1.2}$$

b) $1 \le p \le p' \le q$, $\mu = \dfrac{1}{p'} - \dfrac{1}{q}$, *and* $-\alpha - \dfrac{1}{p'} < \nu \le \dfrac{1}{2} - \dfrac{1}{p'} + \dfrac{1}{q}$.

To prove this Theorem we will use Proposition 3.1 and an interpolation argument of Stein and Weiss. In Appendix 1 we provide examples that show that the bounds for ν in (1.2) are best possible.

Theorem 1.2 *The following inequality holds for every* $1 < p \le 2$, $\alpha \ge -\frac{1}{2}$ *and* $f \in C_0^\infty(\mathbf{R}^+)$

$$\frac{\|L^{\alpha}_{\frac{1}{2},0} f\|_{L^{p'}(\mathbf{R}^+)}}{||f||_{L^p(\mathbf{R}^+)}} \le 2^{\frac{1}{p}-\frac{1}{2}} \frac{p^{\frac{1}{2}\left(\alpha+\frac{1}{2}+\frac{1}{p}\right)}}{(p')^{\frac{1}{2}\left(\alpha+\frac{1}{2}+\frac{1}{p'}\right)}} \frac{\Gamma\left((\alpha+\frac{1}{2})\frac{p'}{2}+\frac{1}{2}\right)^{\frac{1}{p'}}}{\Gamma\left((\alpha+\frac{1}{2})\frac{p}{2}+\frac{1}{2}\right)^{\frac{1}{p}}}. \tag{1.3}$$

The constant on the right-hand side of (1.3) is best possible. The equality in (1.3) is attained by the functions $f_s(x) = x^{\alpha+\frac{1}{2}} e^{-sx^2}$, $s > 0$.

$L^{\alpha}_{\frac{1}{2},0}$ referred to *Hankel transform* in the literature, (see Section 2).

The proof of Theorem 1.2 is a generalization of Beckner's celebrated proof in [Be], and will be performed with a series of steps, some of them crucial, some of them of technical nature. We will write these steps in the form of Lemmas, and, in the process, we will highlight the applications of these Lemmas to other problems in Analysis, the hypercontractivity of the Laguerre semigroup being one of the most significant issues.

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2 The Hankel transforms

Our interest in the class $\mathcal{L}$ was originally motivated by the Fourier transform and the Hankel transform.

The Fourier transform $\hat{f}(\zeta) = \int_{\mathbf{R}^n} e^{-i(x_1\zeta_1+\cdots+x_n\zeta_n)} f(x)\,dx$ is well defined when $f \in C_0^\infty(\mathbf{R}^n)$, and can be extended to a bounded linear operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ if and only if $1 \le p \le 2$ and $q = p'$, (see e.g. [T]). Furthermore,

$$||\hat{f}||_{L^{p'}(\mathbf{R}^n)} \le (2\pi)^{\frac{n}{p'}} \left(p^{\frac{1}{p}}(p')^{-\frac{1}{p'}}\right)^{\frac{n}{2}} ||f||_{L^p(\mathbf{R}^n)}, \qquad f \in C_0^\infty(\mathbf{R}^n). \tag{2.1}$$

The constant on the right-hand side of (2.1) is best possible, as W. Beckner proved in a celebrated paper [Be].

The Gaussian functions $f_s(x) = e^{-s(x_1^2+\cdots+x_n^2)}$, with $s > 0$, attain the equality in (2.1). E. Lieb proved in [L] that the f_s are the only function for which the equality is attained. Since the Fourier transform of a radial function is radial, we can state the following important observation: *the Fourier transform has the same $L^p(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$ norm of its restriction to the radial functions of $L^p(\mathbf{R}^n)$.*

The restriction of the Fourier transform to the space of radial functions can be rewritten as a constant multiple of an operator of the class $\mathcal{L}$. In fact the Fourier transform of $f(|x|)$ is

$$\hat{f}(|\zeta|) = (2\pi)^{\frac{n}{2}} |\zeta|^{-\frac{n}{2}+1} \int_0^{+\infty} f(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(r|\zeta|)\, dr$$

$$= (2\pi)^{\frac{n}{2}} |\zeta|^{-n+1} \int_0^{+\infty} f(r)(|\zeta| r)^{\frac{n}{2}} J_{\frac{n}{2}-1}(r|\zeta|)\, dr = (2\pi)^{\frac{n}{2}} L^{\frac{n}{2}-1}_{\frac{n}{2},\, 1-n} f(|\zeta|). \quad (2.2)$$

Following [CCTV], we will refer to $L^{\alpha}_{\alpha+1,\,-2\alpha-1}$, $\alpha > -1$, as to the *Fourier-Bessel transform* of order α, even though this operator, which H. Hankel introduced in 1875, (see [H]), is sometimes referred to as Hankel transform in the literature. We let

$$\widetilde{\mathcal{H}}_{\alpha} f(x) = L^{\alpha}_{\alpha+1,\,-2\alpha-1} f(x) = \int_0^{+\infty} f(t)(xt)^{-\alpha} J_{\alpha}(xt)\, t^{2\alpha+1} dt. \quad (2.3)$$

From (2.2) follows that

$$\hat{f}(|\zeta|) = (2\pi)^{\frac{n}{2}} \widetilde{\mathcal{H}}_{\frac{n}{2}-1} f(|\zeta|), \quad f \in C_0^{\infty}(\mathbf{R}^+),\ \zeta \in \mathbf{R}^n.$$

The Fourier-Bessel transform of order α shares a lot of properties with the Fourier transform. H. Hankel proved the following inversion formula,

$$\widetilde{\mathcal{H}}_{\alpha} \left(\widetilde{\mathcal{H}}_{\alpha} f \right)(x) = f(x), \qquad f \in C_0^{\infty}(0, +\infty). \quad (2.4)$$

A short and elegant proof of (2.4) is in [CCTV]. It is easy to prove that the Fourier-Bessel transform extends to an isometry on $L^2(\mathbf{R}^+,\ x^{2\alpha+1}dx)$. Moreover, $|\widetilde{\mathcal{H}}_{\alpha} f(x)| \le b_{\alpha} ||f||_{L^1(\mathbf{R}^+,\ t^{2\alpha+1}dt)}$, where $b_{\alpha} = \sup_{t \in \mathbf{R}^+} |t^{-\alpha} J_{\alpha}(t)|$. By Riesz interpolation theorem, the Fourier Bessel transform extends to a bounded linear operator from $L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt)$ to $L^{p'}(\mathbf{R}^+,\, t^{2\alpha+1}dt)$, and

$$\frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}(\mathbf{R}^+,\, t^{2\alpha+1}dt)}}{\|f\|_{L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt)}} \le b_\alpha^{1-\frac{2}{p'}}. \tag{2.5}$$

Note that the $L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt) - L^{p'}(\mathbf{R}^+,\, t^{2\alpha+1}dt)$ norm of $\widetilde{\mathcal{H}}_\alpha$ is the same as the $L^p(\mathbf{R}^+) - L^{p'}(\mathbf{R}^+)$ norm of $L^\alpha_{\frac{2\alpha+1}{p'}-\alpha,\, 0}$. Indeed, if we let $F(t) = t^{\frac{2\alpha+1}{p}} f(t)$, and we observe that $F(t) \in L^p(\mathbf{R}^+)$ if and only if $f(t) \in L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt)$, using (2.3) we can see that

$$\frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}(\mathbf{R}^+,\, t^{2\alpha+1}dt)}}{\|f\|_{L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt)}} = \frac{\|L^\alpha_{\frac{2\alpha+1}{p'}-\alpha,\, 0}F\|_{L^{p'}(\mathbf{R}^+)}}{\|F\|_{L^p(\mathbf{R}^+)}}. \tag{2.6}$$

$L^\alpha_{\frac{1}{2},0}$ is the so-called *Hankel transform of order* α. This is a well studied operator with remarkable properties. We will let

$$\mathcal{H}_\alpha f(x) = L^\alpha_{\frac{1}{2},0} f(x) = \int_0^{+\infty} f(t)(xt)^{\frac{1}{2}} J_\alpha(xt)\, dt. \tag{2.7}$$

The Hankel transform shares many properties with the Fourier transform as well. The following inversion formula for the Hankel transform is proved e.g. in [EMOT]

$$\mathcal{H}_\alpha\left(\mathcal{H}_\alpha f\right)(x) = f(x), \qquad f \in C_0^\infty(0,\, +\infty). \tag{2.8}$$

From (2.8) follows that the Hankel transform extends to an isometry on $L^2(\mathbf{R}^+)$. Moreover, $|\mathcal{H}_\alpha f(x)| \le c_\alpha ||f||_{L^1(\mathbf{R}^+)}$, where $c_\alpha = \sup_{t\in(0,\,+\infty)} |t^{\frac{1}{2}} J_\alpha(t)|$.

By the M. Riesz convexity theorem, the Hankel transform extends to a bounded linear operator from $L^p(\mathbf{R}^+)$ to $L^{p'}(\mathbf{R}^+)$ for every $1 \le p \le 2$, and

$$\frac{||\mathcal{H}_\alpha f||_{L^{p'}(\mathbf{R}^+)}}{||f||_{L^p(\mathbf{R}^+)}} \le c_\alpha^{1-\frac{2}{p'}}. \tag{2.9}$$

In Section 6 we will evaluate the $L^p(\mathbf{R}^+) \to L^{p'}(\mathbf{R}^+)$ norm of this operator. This is one of the main results of this paper, (see Theorem 1.2).

Unfortunately the techniques that we used to compute the $L^p(\mathbf{R}^+) \to L^{p'}(\mathbf{R}^+)$ norm of the Hankel transform cannot be used to compute the $L^p(\mathbf{R}^+,\, t^{2\alpha+1}dt)$ to $L^{p'}(\mathbf{R}^+,\, t^{2\alpha+1}dt)$ norm of the Fourier Bessel transform for arbitrary values of the exponent α. When $\alpha = \frac{n}{2} - 1$, where n is a positive integer, the norm of the Fourier Bessel transform can be computed with the aid of the Theorems of Beckner and Lieb.

The following Proposition will be proved in Appendix 2.

Proposition 2.1 *The following inequality holds for every* $1 < p \le 2$, $n \ge 1$ *and* $f \in C_0^\infty(\mathbf{R}^n)$.

$$\frac{\|\widetilde{\mathcal{H}}_{\frac{n}{2}-1} f\|_{L^{p'}(\mathbf{R}^+,\, r^{n-1}dr)}}{\|f\|_{L^p(\mathbf{R}^+,\, r^{n-1}dr)}} \le \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{p^{\frac{n}{2p}}}{(p')^{\frac{n}{2p'}}} 2^{\left(\frac{n}{2}-1\right)\left(\frac{1}{p'}-\frac{1}{p}\right)}. \tag{2.10}$$

The constant in (2.10) is best possible and coincide with the $L^p(\mathbf{R}^+) - L^{p'}(\mathbf{R}^+)$ *operator norm of* $L^{\frac{n}{2}-1}_{1-\frac{n}{2}+\frac{n-1}{p'},\, 0}$. *The equality in (2.10) is attained by the functions* $f_s(x) = e^{-sx^2}$, $s > 0$.

3 An easy $L^p - L^q$ inequality.

In this section we show that $L^\alpha_{\nu,\,\mu}$ can be viewed as a convolution operator, and we estimate its operator norm using Young inequality for convolution.

Proposition 3.1 $L^\alpha_{\nu,\,\mu}$ *is bounded from* $L^p(\mathbf{R}^+)$ *to* $L^q(\mathbf{R}^+)$ *whenever* $q \ge p \ge 1$, $\alpha > -\frac{1}{2}$, *and* ν *and* μ *satisfy*

$$\mu = \frac{1}{p'} - \frac{1}{q}, \qquad \text{and} \qquad -\alpha - \frac{1}{p'} < \nu < \frac{1}{2} - \frac{1}{p'}. \tag{3.1}$$

For these values of the parameters,

$$\|L^\alpha_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)} \le \|x^{\nu-\frac{1}{q}} J_\alpha\|_{L^r(\mathbf{R}^+)} ||f||_{L^p(\mathbf{R}^+)}, \tag{3.2}$$

where $\dfrac{1}{r} = 1 + \dfrac{1}{q} - \dfrac{1}{p}$.

Proof. Observe, first of all, that homogeneity considerations force $\mu = \dfrac{1}{p'} - \dfrac{1}{q}$.

$L^{\alpha}_{\nu,\,\mu}$ can be viewed as a convolution operator with respect to $\dfrac{dx}{x}$, the Haar measure on $\mathbf{R}^+$. Recall that the convolution of f and g with respect to the measure $\dfrac{dx}{x}$ is $f * g(x) = \displaystyle\int_0^\infty f(t)g(xt)\frac{dt}{t}$.

More specifically,

$$L^{\alpha}_{\nu,\,\mu} f(y) = y^{\mu} \int_0^\infty (xy)^{\nu} f(x) J_\alpha(xy) dx$$

$$= y^{\mu} \int_0^\infty (xy)^{\nu} x^{\frac{1}{p'}} x^{\frac{1}{p}} f(x) J_\alpha(xy) \frac{dx}{x}$$

$$= y^{\mu - \frac{1}{p'}} \int_0^\infty (xy)^{\nu + \frac{1}{p'}} x^{\frac{1}{p}} f(x) J_\alpha(xy) \frac{dx}{x} = y^{\mu - \frac{1}{p'}} \left(x^{\frac{1}{p}} f(x) * J_\alpha(x)(x)^{\nu + \frac{1}{p'}} \right).$$

Since $\mu - \dfrac{1}{p'} = -\dfrac{1}{q}$, if we let $F(x) = x^{\frac{1}{p}} f(x)$ and $G(x) = x^{\nu + \frac{1}{p'}} J_\alpha(x)$, we can conclude that $L^{\alpha}_{\nu,\,\mu} f(y) = y^{-\frac{1}{q}} \left(F * G\right)(y)$. Therefore, the inequality $||L^{\alpha}_{\nu,\,\mu} f||_{L^q(\mathbf{R}^+)} \leq C||f||_{L^p(\mathbf{R}^+)}$ is equivalent to $||F*G||_{L^q(\mathbf{R}^+,\, \frac{dx}{x})} \leq C||F||_{L^p(\mathbf{R}^+,\, \frac{dx}{x})}$. By Young's inequality for convolution,

$$||F * G||_{L^q(\mathbf{R}^+,\, \frac{dx}{x})} \leq ||F||_{L^p(\mathbf{R}^+,\, \frac{dx}{x})} ||G||_{L^r(\mathbf{R}^+,\, \frac{dx}{x})}$$

provided that $1 + \dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{r}$ and $G(x) = x^{\nu + \frac{1}{p'}} J_\alpha(x) \in L^r(\mathbf{R}^+,\, \dfrac{dx}{x})$.

Since $|x^{\nu + \frac{1}{p'}} J_\alpha(x)| = \mathcal{O}(x^{\alpha + \nu + \frac{1}{p'}})$ as $x \to 0$, and $|x^{\nu + \frac{1}{p'}} J_\alpha(x)| = \mathcal{O}(x^{\nu + \frac{1}{p'} - \frac{1}{2}})$ as $x \to \infty$, we can see that $x^{\nu + \frac{1}{p'}} J_\alpha(x) \in L^r(\mathbf{R}^+,\, \dfrac{dx}{x})$, (or equivalently, $x^{\nu - \frac{1}{q}} J_\alpha(x) \in L^r(\mathbf{R}^+)$), if and only if

$$\alpha + \nu + \frac{1}{p'} > 0 \quad \text{and} \quad \nu + \frac{1}{p'} - \frac{1}{2} < 0,$$

or $-\alpha - \frac{1}{p'} < \nu < \frac{1}{2} - \frac{1}{p'}$, as in (3.1).

Remark 1. Note that (3.1) forces $\alpha > -\frac{1}{2}$.

Remark 2. The operator $L^{\alpha}_{\frac{1}{2},-1}$ is bounded from $L^1(\mathbf{R}^+)$ to $L^{1,\infty}(\mathbf{R}^+)$. Indeed,

$$L^{\alpha}_{\frac{1}{2},-1} f(y) = y^{-1} \int_0^\infty (xy)^{\frac{1}{2}} J_\alpha(xy) f(x) dx = y^{-1} \mathcal{H}_\alpha f(y)$$

is in $L^{1,\infty}(\mathbf{R}^+)$ because $\mathcal{H}_\alpha f(y)$ is bounded whenever $f \in L^1(\mathbf{R}^+)$, and

$$\|L^\alpha_{\frac{1}{2},\,-1} f\|_{L^{1,\infty}(\mathbf{R}^+)} \le \|\mathcal{H}_\alpha f\|_{L^\infty(\mathbf{R}^+)} \le c_\alpha \|f\|_{L^1(\mathbf{R}^+)}, \tag{3.3}$$

where $c_\alpha = \sup_{t\in(0,+\infty)} |t^{\frac{1}{2}} J_\alpha(t)|$.

4 Open problems and conjectures.

Proposition 3.1 is not optimal in many respects. First of all, the range of values for ν and α in (3.1) is not optimal, since it does not include $\nu = \frac{1}{2}$ (as in the Hankel transform!) and $\alpha = -\frac{1}{2}$. Also, the inequality (3.2) is not sharp.

The problem of evaluating the $L^p - L^q$ operator norm of $L^\alpha_{\nu,\,\mu}$ for general values of the parameter seems to be very difficult. At the moment we can only evaluate it when $q = p'$, $\alpha \ge -\frac{1}{2}$ and $\nu = \frac{1}{2}$, (Theorem 1.2), and when $q = p'$, $\alpha = \dfrac{n}{2} - 1$, where $n \ge 2$ is an integer, and $\nu = 1 - \dfrac{n}{2} + \dfrac{n-1}{p'}$, (Proposition 2.1)

The theorem of E. Lieb implies that the Gaussian functions are the only maximizers for operator norm of the Fourier-Bessel transform when $\alpha = \frac{n}{2} - 1$. We conjecture that this is true in general, that is, that the Gaussian functions are the only maximizers of the ratio $\dfrac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}(\mathbf{R}^+,\, r^{n-1}dr)}}{\|\widetilde{f}\|_{L^p(\mathbf{R}^+,\, r^{n-1}dr)}}$ for every $\alpha \ge -\frac{1}{2}$. If that is the case, then the $L^p(\mathbf{R}^+,\, r^{2\alpha+1}dr) - L^{p'}(\mathbf{R}^+,\, r^{2\alpha+1}dr)$ norm of the Hankel transform is

$$\sup \frac{\|\widetilde{\mathcal{H}}_\alpha f\|_{L^{p'}(\mathbf{R}^+,\, r^{2\alpha+1}dr)}}{\|\widetilde{f}\|_{L^p(\mathbf{R}^+,\, r^{2\alpha+1}dr)}} = \sup \frac{\|L^\alpha_{\frac{2\alpha+1}{p'}-\alpha,\ 0}(F)\|_{L^{p'}(\mathbf{R}^+)}}{\|F\|_{L^p(\mathbf{R}^+)}}$$

$$= 2^{\alpha\left(\frac{1}{p'}-\frac{1}{p}\right)} p^{\frac{\alpha+1}{p}} (p')^{-\frac{\alpha+1}{p'}} \Gamma(\alpha+1)^{\frac{1}{p'}-\frac{1}{p'}}. \tag{4.1}$$

More in general, we conjecture the following:

Conjecture 1. *The $L^p \to L^q$ norm of $L^\alpha_{\nu,\mu}$ is finite if $\alpha \ge -\frac{1}{2}$, $1 \le p \le q \le \infty$, and if and only if ν and μ are such that*

$$\mu = \frac{1}{p'} - \frac{1}{q}, \qquad \textit{and} \qquad -\alpha - \frac{1}{p'} < \nu \le \frac{1}{2}. \tag{4.2}$$

For these values of the parameters,

$$\sup \frac{\|L^\alpha_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)}}{\|f\|_{L^p(\mathbf{R}^+)}} = C^\alpha_{\nu,p,q}, \tag{4.3}$$

where we have let

$$C^\alpha_{\nu,p,q} = 2^{\nu-\frac{1}{q}} \frac{p^{\frac{1}{2}(1+\alpha-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\alpha+\nu+\frac{1}{p'})}} \frac{\Gamma\left(\frac{\alpha+\nu+\mu}{2}q+\frac{1}{2}\right)^{\frac{1}{q}}}{\Gamma\left(\frac{1+\alpha-\nu}{2}p+\frac{1}{2}\right)^{\frac{1}{p}}}. \tag{4.4}$$

Conjecture 2. *The functions $f_s(x) = e^{-sx^2}x^{1-\nu+\alpha}$, $s > 0$, are the only maximizers of the ratio in (4.3).*

Both conjectures hold true in special and significant cases, (see Proposition 2.1 and Theorem 1.2). We have proved in Appendix 1 that the bounds for ν are optimal.

Proving that $\dfrac{\|L^\alpha_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)}}{\|f\|_{L^p(\mathbf{R}^+)}} \geq C^\alpha_{\nu,p,q}$ is easy. The functions $f_s(x)$ defined above are in $L^p(\mathbf{R}^+)$ because, by (4.2) and the fact that $\alpha \geq -\frac{1}{2}$, $1-\nu+\alpha \geq \dfrac{1}{2}+\alpha \geq 0 > -\dfrac{1}{p}$. A change of variables shows that the ratio $\dfrac{\|L^\alpha_{\nu,\mu}(f_s)\|_{L^q(\mathbf{R}^+)}}{\|f_s\|_{L^p(\mathbf{R}^+)}}$ is independent of s. When $s=\frac{1}{2}$, $L^\alpha_{\nu,\mu}(f_{\frac{1}{2}})(y) = y^\mu \displaystyle\int_0^\infty e^{-\frac{x^2}{2}} J_\alpha(xy) x^{\alpha+1} dx$ can be explicitly computed, (see e.g. [EMOT], pg. 29, n. 10), and is $y^{\alpha+\nu+\mu}e^{-\frac{y^2}{2}}$. Thus, by the well known identity

$$\int_0^\infty e^{-sx^2}x^m dx = \frac{s^{-\frac{1+m}{2}}}{2}\Gamma\left(\frac{1+m}{2}\right), \quad m > -1, \tag{4.5}$$

follows that

$$\frac{\|L^\alpha_{\nu,\mu}(f_s)\|_{L^q(\mathbf{R}^+)}}{\|f_s\|_{L^p(\mathbf{R}^+)}} = \frac{\|y^{\nu+\alpha+\mu}e^{-\frac{y^2}{2}}\|_{L^q(\mathbf{R}^+)}}{\|x^{1-\nu+\alpha}e^{-\frac{x^2}{2}}\|_{L^p(\mathbf{R}^+)}}$$

$$= \frac{\left(\int_0^\infty y^{(\nu+\alpha+\mu)q} e^{-\frac{qy^2}{2}}\right)^{\frac{1}{q}}}{\left(\int_0^\infty x^{(1-\nu+\alpha)p} e^{-\frac{px^2}{2}}\right)^{\frac{1}{p}}} = 2^{\nu-\frac{1}{q}} \frac{p^{\frac{1}{2}(1+\alpha-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\alpha+\nu+\frac{1}{p'})}} \frac{\Gamma\left(\frac{\alpha+\nu+\mu}{2}\, q+\frac{1}{2}\right)^{\frac{1}{q}}}{\Gamma\left(\frac{1+\alpha-\nu}{2}\, p+\frac{1}{2}\right)^{\frac{1}{p}}} = C^{\alpha}_{\nu,\,p,\,q}.$$

5. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on a Theorem on interpolation of operators with change of measure proved by E. Stein and G. Weiss in [SW].

Let $(M,\, \mathcal{M}, d\alpha)$ and $(N,\, \mathcal{N}, d\mu)$ be measure spaces. Let T be a sublinear operator mapping a class of functions on M into a class of functions on N.

Let $d\beta_i$ $i=0,\, 1$ be measures on $\mathcal{M}$, and let $d\alpha_i$ $i=0,\, 1$ be measures on $\mathcal{N}$. We let $\beta=\beta_0+\beta_1$, $\alpha=\alpha_0+\alpha_1$. By Radon-Nikodim theorem there exist functions $h_i(x)$ on $\mathcal{M}$ and $k_i(y)$ on $\mathcal{N}$ which are such that, for every $d\beta_i$-measurable subset of $\mathcal{M}$ and every $d\alpha_i$ measurable subset of $\mathcal{N}$,

$$\beta_i(E)=\int_E h_i(y)d\beta(y), \quad \alpha_i(F)=\int_F k_i(x)d\alpha(x).$$

For every $r,\, s\in[0,1]$ we can define the following measures on $\mathcal{M}$ and $\mathcal{N}$

$$\beta_s(E)=\int_E h_1^s h_0^{1-s}(x)d\beta(x), \quad \alpha_r(F)=\int_F k_1^r k_0^{1-r}(x)d\alpha(x).$$

So, if we let $\beta_i=y^{m_i}dy$ and $\alpha_i=x^{n_i}dx$ for some $m_i,\, n_i\in\mathbf{R}$, we gather

$$\beta_s=y^{sm_1+(1-s)m_0}dy, \quad \alpha_r=x^{rn_1+(1-r)n_0}dx.$$

Let $1\le p_0\ne p_1\le\infty$, $1\le q_0\ne q_1\le\infty$. For every $t\in(0,1)$, we consider the exponents q_t and p_t that satisfy the following relation.

$$\frac{1}{p_t}=\frac{t}{p_1}+\frac{1-t}{p_0}, \quad \text{and} \quad \frac{1}{q_t}=\frac{t}{q_1}+\frac{1-t}{q_0}.$$

We also let $s(t)=\dfrac{tq_t}{q_1}$, (so that $1-s(t)=(1-t)\dfrac{q_t}{q_0}$) and $r(t)=\dfrac{tp_t}{p_1}$. With this position

$$\beta_{s(t)}=y^{q_t\left(t\frac{m_1}{q_1}+(1-t)\frac{m_0}{q_0}\right)}dy, \quad \text{and} \quad \alpha_{r(t)}=x^{p_t\left(t\frac{n_1}{p_1}+(1-t)\frac{n_0}{p_0}\right)}dx.$$

The main theorem in [SW] can be stated as follows.

Theorem 5.1 *Let T be a sublinear operator satisfying*

$$||Tf||_{L^{p_i}(M,d\beta_i)} \leq K_i||f||_{L^{q_i}(N,d\alpha_i)}$$

(or $||Tf||_{L^{p_i,\infty}(M,d\beta_i)} \leq K_i||f||_{L^{q_i}(N,d\alpha_i)}$), for every $f \in L^p(M,\ \mathcal{M},\ d\beta_i)$ and $i = 0,\ 1,$. Then T is defined also in $L^p(M, d\beta_{s(t)})$ for every $t \in (0,1)$, and

$$||Tf||_{L^p(M,d\beta_{s(t)})} \leq K_t||f||_{L^q(N,d\alpha_{r(t)})}.$$

where K_t is independent of f.

To prove Theorem 1.1 we argue as follows: When $q \leq p'$ and $\nu = \frac{1}{2}$, we let

$$TF(y) = \int_0^\infty F(x)J_\alpha(xy)dx, \quad F \in C_0^\infty(\mathbf{R}).$$

Then

$$L^\alpha_{\frac{1}{2},\mu}f(y) = y^{\mu+\frac{1}{2}}\int_0^\infty x^{\frac{1}{2}}f(x)J_\alpha(xy)dx = y^{\mu+\frac{1}{2}}T(x^{\frac{1}{2}}f)(y).$$

If we let $d\beta = y^{q(\mu+\frac{1}{2})}dy$, $d\alpha = x^{-\frac{p}{2}}dx$ and $F(x) = x^{\frac{1}{2}}f(x)$, we can see at once that the inequality

$$\|TF\|_{L^q(\mathbf{R}^+,\ d\beta)} \leq C\|F\|_{L^p(\mathbf{R}^+,\ d\alpha)},$$

(resp. $\|TF\|_{L^{q,\infty}(\mathbf{R}^+,\ d\beta)} \leq C\|F\|_{L^p(\mathbf{R}^+,\ d\alpha)}$), is equivalent to

$$\|L^\alpha_{\alpha,\mu}f\|_{L^q(\mathbf{R}^+)} \leq C\|f\|_{L^p(\mathbf{R}^+)},$$

(resp. $\|L^\alpha_{\alpha,\mu}f\|_{L^{q,\infty}(\mathbf{R}^+)} \leq C\|f\|_{L^p(\mathbf{R}^+)}$).

When $q = p'$ we are in the case of the Hankel transform, so we assume $q < p'$. The point $\left(\frac{1}{p},\ \frac{1}{q}\right)$ is above the duality line $\dfrac{1}{q} = 1 - \dfrac{1}{p}$. We apply the Theorem of Stein and Weiss with $p_1 = q_1 = 1$ and $q_0 = p_0'$ that satisfy

$$\frac{1}{p} = \frac{1-t}{p_0} + t, \quad \frac{1}{q} = \frac{1-t}{p_0'} + t$$

for some $t \in (0,1)$. The point $\left(\frac{1}{p_0},\ \frac{1}{p_0'}\right)$ is the intersection of the line in $[0,1] \times [0,1]$ that joins the points $\left(\frac{1}{p},\ \frac{1}{q}\right)$ and $(1,1)$, and the duality line.

We gather

$$t = \frac{1}{q} - \frac{1}{p'}, \qquad p_0 = 1 + \frac{q'}{p'}. \tag{5.1}$$

By (3.3)

$$||L^{\alpha}_{\frac{1}{2},-1} f||_{L^{1,\infty}(\mathbf{R}^+)} \le c_\alpha \|f\|_{L^1(\mathbf{R}^+)},$$

and by (4.1)

$$||L^{\alpha}_{\frac{1}{2},0} f||_{L^{p'_0}(\mathbf{R}^+)} \le C^{\alpha}_{\frac{1}{2},p_0,p'_0} \|f\|_{L^{p_0}(\mathbf{R}^+)}.$$

Therefore,

$$||TF||_{L^{1,\infty}(\mathbf{R}^+,\ d\beta_1)} \le c_\alpha \|F\|_{L^1(\mathbf{R}^+,\ d\alpha_1)},$$

and

$$||TF||_{L^{p'_0}(\mathbf{R}^+, d\beta_0)} \le C^{\alpha}_{\frac{1}{2},p_0,p'_0} \|F\|_{L^{p_0}(\mathbf{R}^+, d\alpha_0)},$$

where we have let

$$d\beta_1 = y^{-\frac{1}{2}} dy,\ d\beta_0 = y^{\frac{p'_0}{2}} dy,\ d\alpha_1 = x^{-\frac{1}{2}} dx,\ d\alpha_0 = x^{-\frac{p}{2}} dx.$$

We can apply the theorem of Stein and Weiss and conclude that

$$||TF||_{L^q(\mathbf{R}^+, d\beta_{s(t)})} \le C_t \|F\|_{L^p(\mathbf{R}^+, d\alpha_{r(t)})}, \tag{5.2}$$

where C_t does not depend on f. By the definitions of $d\beta_{s(t)}$, $d\alpha_{r(t)}$ and (5.1),

$$d\beta_{s(t)} = y^{q\left(-\frac{t}{2} + \frac{(1-t)p'_0}{2p'_0}\right)} dy = y^{q\left(\frac{1}{2}+\mu\right)} dy$$

and

$$d\alpha_{r(t)} = x^{-p\left(-\frac{t}{2} - \frac{(1-t)p_0}{2p_0}\right)} dx = x^{-\frac{p}{2}} dx,$$

and so (5.2) is equivalent to

$$||L^{\alpha}_{\frac{1}{2},\mu} f||_{L^q(\mathbf{R}^+)} \le C_t \|f\|_{L^p(\mathbf{R}^+)}, \tag{5.3}$$

as required.

To prove the Theorem for $q \le p'$ and $\dfrac{1}{2} - \dfrac{1}{p'} \le \nu < \dfrac{1}{2}$ we let $\nu = \frac{1}{2} - \epsilon$, with $0 < \epsilon \le \dfrac{1}{p'}$. We use the identity (6.1) in the Appendix 1 and the inversion formula for the Hankel transform to conclude that $(xy)^{\frac{1}{2}-\epsilon} J_\alpha(xy) =$

$\mathcal{H}_{\alpha-\epsilon}\psi_\epsilon(xy)$, where we have let $\psi_\epsilon(t) = 2^{1-\epsilon}\Gamma(\epsilon)^{-1}\chi_{(0,1)}(t)(1-t^2)^{\epsilon-1}t^{\alpha-\epsilon+\frac{1}{2}}$. With this position

$$L^\alpha_{\nu,\mu}f(y) = y^\mu \int_0^\infty f(x)\mathcal{H}_{\alpha-\epsilon}\psi_\epsilon(xy)dx = y^\mu \int_0^\infty \psi_\epsilon(z)\mathcal{H}_{\alpha-\epsilon}f(zy)dz$$

$$= \int_0^\infty \psi_\epsilon(z) L^{\alpha-\epsilon}_{\frac{1}{2},\mu}\psi_\epsilon(zy)dz.$$

Thus, by (5.3),

$$||L^\alpha_{\nu,\mu}f||_{L^q(\mathbf{R}^+)} \leq \int_0^\infty \psi_\epsilon(z)||L^{\alpha-\epsilon}_{\frac{1}{2},\mu}\psi_\epsilon(z\,.)||_{L^q(\mathbf{R}^+)}dz$$

$$\leq C_t||f||_{L^p(\mathbf{R}^+)}\int_0^\infty z^{-\frac{1}{q}}\psi_\epsilon(z)dz$$

$$= 2^{1-\epsilon}\Gamma(\epsilon)^{-1}C_t||f||_{L^p(\mathbf{R}^+)}\int_0^1 (1-z^2)^{\epsilon-1}z^{\alpha-\epsilon+\frac{1}{2}-\frac{1}{q}}dz.$$

The integral is finite because $\alpha \geq -\frac{1}{2}$, and

$$\alpha - \epsilon + \frac{1}{2} - \frac{1}{q} > -\epsilon - \frac{1}{q} > -\frac{1}{p'} - \frac{1}{q} > -1$$

since we have assumed $p < q < p'$. This concludes the proof of part a) of the Theorem 1.1.

To prove part b) we observe that the adjoint of $L^\alpha_{\nu,\mu}$ is $L^\alpha_{\nu+\mu,\,-\mu}$. Since $q > p'$, by part a) the $L^{q'} - L^{p'}$ norm of $L^\alpha_{\nu+\mu,\,-\mu}$ is finite is $\nu + \mu \leq \frac{1}{2}$, that is, $\nu \leq \frac{1}{2} - \frac{1}{p'} - \frac{1}{q}$. By duality, the $L^p - L^q$ norm of $L^\alpha_{\nu,\mu}$ is finite as well. This concludes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

The proof of Theorem 1.2 is a generalization of Beckner's celebrated proof in [Be], and will be performed with a series of steps which are important in their own because of the connection with other problems in Analysis, the hypercontractivity of the Laguerre semigroup being one of the most significant issues.

6.1 Preliminaries

In this section we collect together a few preliminary facts concerning the Laguerre polynomials and we state our main Theorem. We refer to [Sz] or to [Th] for details.

For $\alpha > -1$, $x > 0$ and $k = 0,\, 1,\, 2, \cdots$, the *Laguerre polynomials of type* α are defined by the formula

$$e^{-x} x^{\alpha} L_k^{\alpha}(x) = \frac{1}{k!} \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}). \tag{6.1}$$

Each L_k^{α} is a polynomial of degree k. The Laguerre polynomials satisfy the following orthogonality relations,

$$\int_0^{+\infty} L_k^{\alpha}(x) L_j^{\alpha}(x) e^{-x} x^{\alpha} dx = \begin{cases} 0, & \text{if } k \neq j, \\ \dfrac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}, & \text{if k=j.} \end{cases} \tag{6.2}$$

A change of variables shows that the orthogonality relation (6.2) can be rewritten as

$$\int_{-\infty}^{+\infty} L_k^{\alpha}(x^2) L_j^{\alpha}(x^2) e^{-x^2} x^{2\alpha+1} dx = \begin{cases} 0, & \text{if } k \neq j, \\ \dfrac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}, & \text{if k=j.} \end{cases} \tag{6.3}$$

The polynomials $L_k^{\alpha}(x^2)$ are a constant multiple of the so-called *Generalized Hermite polynomials* of order $\alpha + \frac{1}{2}$ and even degree. See [Ro].

We shall use the following important identity, often called *the Hille-Hardy identity*, which is valid for real or complex ω's such that $|\omega| < 1$ and for $x,\ y \in \mathbf{R}$.

$$K_{\omega}^{\alpha}(x^2,\, y^2) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} L_k^{\alpha}(x^2) L_k^{\alpha}(y^2) \omega^k$$

$$= (1-\omega)^{-1} (xy)^{-\alpha} (-\omega)^{-\frac{\alpha}{2}} e^{-\frac{\omega}{1-\omega}(x^2+y^2)} J_{\alpha} \left(\frac{2xy(-\omega)^{\frac{1}{2}}}{1-\omega} \right). \tag{6.4}$$

$K_{\omega}^{\alpha}(x^2,\, y^2)$ is the *Mehler kernel of order* α. By (6.3),

$$\int_{-\infty}^{\infty} K_{\omega}^{\alpha}(x^2,\, y^2) L_k^{\alpha}(x^2)\, e^{-x^2} |x|^{2\alpha+1} dx = \omega^k L_k^{\alpha}(y^2). \tag{6.5}$$

In what follows we will let

$$T_\omega^\alpha(\psi)(t) = \int_{-\infty}^{\infty} K_\omega^\alpha(x^2, t^2)\psi(x)|x|^{2\alpha+1}e^{-x^2}dx, \tag{6.6}$$

where $|\omega| < 1$ is a complex parameter. This is, up to a constant of normalization and change of variables, the Laguerre semigroup. See e.g. [Th].

6.2 Hypercontractivity of the Laguerre semigroup

The Hille-Hardy identity allows us to replace the Bessel function in $L^\alpha_{\nu,\mu}$ with the Mehler kernel, and to establish a connection between the $L^p - L^q$ mapping properties of these operators and the hypercontractivity of the Laguerre semigroup. We prove the following

Lemma 6.1 *Let $1 \le p \le q \le \infty$, $\alpha \ge -\frac{1}{2}$, and μ and ν such that*

$$\mu = \frac{1}{p'} - \frac{1}{q}, \qquad \textit{and} \qquad -\alpha - \frac{1}{p'} < \nu \le \frac{1}{2}, \tag{6.7}$$

(see(4.2)). We let

$$q(\alpha+\nu+\mu) = 2\gamma+1, \quad (\alpha-\nu+1)p = 2\beta+1, \tag{6.8}$$

and $c_m = 2^{m+1}\Gamma(m+1) = \int_{\mathbf{R}} e^{-\frac{x^2}{2}} x^{2m+1}dx$. Let $C^\alpha_{\nu,p,q}$ be defined as in (4.4).

The inequality

$$\|L^\alpha_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)} \le C^\alpha_{\nu,p,q}\|f\|_{L^p(\mathbf{R}^+)} \tag{6.9}$$

is valid for every $f \in C_0^\infty(\mathbf{R}^+)$ if and only if the inequality

$$\left(\int_{\mathbf{R}} |T^\alpha_{\overline{\omega}}(k)(y\epsilon)|^{p'} \frac{e^{-\frac{y^2}{2}}|y|^{2\gamma+1}dy}{c_\gamma}\right)^{\frac{1}{q}} \le \left(\int_{\mathbf{R}} |k(x\epsilon)|^p \frac{e^{-\frac{x^2}{2}}|x|^{2\beta+1}dx}{c_\beta}\right)^{\frac{1}{p}} \tag{6.10}$$

where $\epsilon = \sqrt{\dfrac{p+q}{2pq}}$, is valid for every even polynomial k and for $\overline{\omega} = -pq^{-1}$.

Remark. Note that in the case of the Fourier Bessel transform $\nu = \frac{2\alpha+1}{p'} - \alpha$ and $\mu = 0$, and so $2\beta+1 = 2\gamma+1 = 2\alpha+1$. In the case of the Hankel transform, $\nu = \frac{1}{2}$ and $\mu = 0$, and $\frac{2\beta+1}{p} = \frac{2\gamma+1}{q}$. Observe also that the assumptions on μ and ν in (6.7) imply

$$-\frac{1}{q} < \frac{2\gamma+1}{q} \leq \frac{2\beta+1}{p} + \mu. \tag{6.11}$$

Proof. It is convenient to use the new notation and rewrite (6.9) in the following fashion.

$$\|L^{\alpha}_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)} \leq 2^{\frac{1}{p}-\frac{1}{q}} \frac{(c_\gamma (q)^{-\gamma-1})^{\frac{1}{q}}}{(c_\beta p^{-\beta-1})^{\frac{1}{p}}} \|f\|_{L^p(\mathbf{R}^+)}. \tag{6.12}$$

Since the functions of the form of $f(x) = x^{\alpha-\nu+1} e^{-\frac{x^2}{2}} k(x)$, (or, with the new notation, $f(x) = x^{\frac{2\beta+1}{p}} e^{-\frac{x^2}{2}} k(x)$), where k is an even polynomial), are dense in $L^p(\mathbf{R}^+)$ when $p \leq 2$, (see e.g. [AW]), it is enough to prove (6.9) for these functions.

Then, we use the Hille-Hardy identity to replace $J_\alpha(xy)$ with the product of an exponential function and the Mehler kernel of order α. Indeed, if we let $x = Ax_1$, $y = B\, y_1$, with

$$A = \sqrt{\frac{p+q}{2q}}, \quad B = \sqrt{\frac{p+q}{2p}},$$

and we let $\varpi = -\frac{p}{q}$, so that

$$\frac{2AB(-\varpi)^{\frac{1}{2}}}{1-\varpi} = 1,$$

we can write

$$J_\alpha\left(\frac{2xy(-\varpi)^{\frac{1}{2}}}{1-\varpi}\right) = J_\alpha\left(\frac{2ABx_1y_1(-\varpi)^{\frac{1}{2}}}{1-\varpi}\right) = J_\alpha(x_1y_1)$$

$$= (1-\varpi)\left(-\varpi(ABx_1y_1)^2\right)^{\frac{\alpha}{2}} e^{\frac{\varpi}{1-\varpi}((Ax_1)^2+(By_1)^2)} K^{\alpha}_{\varpi}((Ax_1)^2,\ (By_1)^2)$$

$$= 2\left(\frac{p+q}{2q}\right)^{\alpha+1} (x_1y_1)^\alpha e^{-\frac{1}{2}(y_1^2+\frac{p}{q}x_1^2)} K^{\alpha}_{\varpi}((Ax_1)^2,\ (By_1)^2).$$

Recalling that $f(x) = e^{-\frac{x^2}{2}} x^{\alpha-\nu+1} k(x)$, where $k(x)$ is an even polynomial, we gather

$$L^{\alpha}_{\nu,\mu} f(y_1) = y_1^{\mu} \int_0^{\infty} (x_1 y_1)^{\nu} J_{\alpha}(x_1 y_1) e^{-\frac{x_1^2}{2}} x_1^{\alpha-\nu+1} k(x_1) dx_1$$

$$= \left(\frac{p+q}{2q}\right)^{\alpha+1} y_1^{\alpha+\nu} e^{-\frac{1}{2} y_1^2}$$

$$\times \int_{-\infty}^{\infty} |x_1|^{2\alpha+1} k(x_1) e^{-\frac{p+q}{2q} x_1^2} K^{\alpha}_{\omega}((Ax_1)^2,\ (By_1)^2) dx_1.$$

If we let $Ax_1 = \sqrt{\frac{p+q}{2q}} x_1 = x$ in the integral above, and we let $\delta_{A^{-1}} k(x) = k(A^{-1}x)$, we can write

$$L^{\alpha}_{\nu,\mu} f(y_1) = y_1^{\alpha+\nu} e^{-\frac{1}{2} y_1^2} \int_{-\infty}^{\infty} k(A^{-1}x) e^{-x^2} K^{\alpha}_{\omega}(x^2,\ (By_1)^2) |x|^{2\alpha+1} dx$$

$$= y_1^{\alpha+\nu+\mu} e^{-\frac{1}{2} y_1^2} T^{\alpha}_{\omega}(\delta_{A^{-1}} k)(By_1), \tag{6.13}$$

where $T^{\alpha}_{\omega}(\psi)(y) = \int_{-\infty}^{\infty} K^{\alpha}_{\omega}(x^2, y^2) \psi(x) |x|^{2\alpha+1} e^{-x^2} dx$ is as in (6.6).

From (6.12), (6.13) and the fact that $\alpha + \nu + \mu = 2\gamma + 1$, follows that the inequality

$$\|L^{\alpha}_{\nu,\mu} f\|_{L^q(\mathbf{R}^+)} \leq C^{\alpha}_{\nu,p,q} \|f\|_{L^p(\mathbf{R}^+)}$$

is equivalent to

$$\left(\int_{-\infty}^{\infty} \frac{|y_1|^{2\gamma+1}}{c_\gamma} e^{-\frac{q}{2} y_1^2} |T^{\alpha}_{\omega}(\delta_{A^{-1}} k)(By_1)|^q dy_1\right)^{\frac{1}{q}}$$

$$\leq \frac{p^{\beta+1}}{q^{\gamma+1}} \left(\int_{-\infty}^{\infty} \frac{|x|^{2\beta+1}}{c_\beta} e^{-\frac{p}{2} x^2} |k(x)|^p dx\right)^{\frac{1}{p}}. \tag{6.14}$$

We let $y_1 \sqrt{q} = y$ in the first integral and $x\sqrt{p} = t$ in the first integral. With this substitutions, the inequality in (6.14) reduces to

$$\left(\int_{\mathbf{R}} |T^{\alpha}_{\omega}(\delta_{A^{-1}} k)(q^{-\frac{1}{2}} By)|^q \frac{e^{-\frac{y^2}{2}} |y|^{2\gamma+1} dy}{c_\gamma}\right)^{\frac{1}{q}} \leq \left(\int_{\mathbf{R}} |k(p^{-\frac{1}{2}} t)|^p \frac{e^{-\frac{t^2}{2}} |t|^{2\beta+1} dt}{c_\beta}\right)^{\frac{1}{p}}$$

Since $q^{-\frac{1}{2}}B = \sqrt{\frac{p+q}{2qp}} = p^{-\frac{1}{2}}A$, we can let $\varepsilon = \sqrt{\frac{p+q}{2qp}}$, and write

$$\left(\int_{\mathbf{R}} |T^{\alpha}_{\omega}(\delta_{A^{-1}}k)(\varepsilon y)|^q \frac{e^{-\frac{y^2}{2}}|y|^{2\gamma+1}}{c_\gamma} dy\right)^{\frac{1}{q}}$$

$$\leq \left(\int_{\mathbf{R}} |\delta_{A^{-1}}k(\varepsilon t)|^p \frac{e^{-\frac{t^2}{2}}|t|^{2\beta+1}}{c_\beta} dt\right)^{\frac{1}{p}} \tag{6.15}$$

as required.

6.3 Reduction to a discrete operator.

In the next crucial step we will approximate the measures $\frac{e^{-\frac{t^2}{2}}|t|^{2\beta+1}}{c_\beta}dx$ and $\frac{e^{-\frac{y^2}{2}}|y|^{2\gamma+1}}{c_\gamma}dy$ that appear in (6.15) with sequences of discrete measures and the generalized Hermite polynomials with homogeneous functions in n variables. Then we will define a discrete analogue of T^{α}_{ω} and we will show that Lemma 6.1 follows from the establishment of the $L^p - L^q$ mapping properties of this operator.

In these reductions we will ignore the factor ε that appears in (6.10) since it can be handled with some extra technicality, and we will replace $\delta_{A^{-1}}k$ with k without loss of generality.

Before we state the following Lemma we need some preliminaries. Let δ_{t_0} be the Dirac distribution on $\mathbf{R}$ with unitary mass at t_0. For every positive integer n we let $d\nu(t)$ be the Bernoulli trial $\frac{1}{2}(\delta_1(t) + \delta_{-1}(t))$. For every integer $n \geq 1$, we let $\bar{t} = (t_1, ..., t_n)$, $d\nu_n(\bar{t}) = d\nu_1(\sqrt{n}\, t_1) \dots d\nu_1(\sqrt{n}\, t_n)$, and $\sigma(\bar{t}) = t_1 + \dots + t_n$.

Let $\psi_{k,n}(\bar{t}) = k! \sum_{1 \leq m_1 < \cdots < m_k \leq n} t_{m_1} \cdots t_{m_k}$ be the elementary symmetric function in n variables of degree k, and let $\mathbf{X_n}$ be the vector space which is spanned by the functions $\sigma(\bar{t})^j \psi_{k,n}(\bar{t})$, with $m \in \mathbf{N}$ and $j \leq 2m$. Note that there functions are homogeneous of degree $k + j$.

We prove the following

Lemma 6.2 *Let T_ω^α be defined as in (6.6). If there exists $N > 0$ which is such that the inequality*

$$\left(\int_{\mathbf{R}^n} |g(\sqrt{\omega}\,\overline{s})|^q \frac{|\sigma(\overline{s})|^{2\gamma+1}}{(2\pi)^{-\frac{1}{2}} c_\gamma} d\nu_n(\overline{s})\right)^{\frac{1}{q}} \leq \left(\int_{\mathbf{R}^n} |g(\overline{s})|^p \frac{|\sigma(\overline{s})|^{2\beta+1}}{(2\pi)^{-\frac{1}{2}} c_\beta} d\nu_n(\overline{s})\right)^{\frac{1}{p}} \tag{6.16}$$

is valid for every $n > N$ and for every function $g(\overline{s}) \in \mathbf{X_n}$, then the inequality

$$\left(\int_{\mathbf{R}} |T_\omega^\alpha(k)(s)|^q \frac{e^{-\frac{s^2}{2}} |s|^{2\gamma+1} ds}{c_\gamma}\right)^{\frac{1}{q}} \leq \left(\int_{\mathbf{R}} |k(s)|^p \frac{e^{-\frac{s^2}{2}} |s|^{2\beta+1} ds}{c_\beta}\right)^{\frac{1}{p}} \tag{6.17}$$

is valid for every even polynomial $k(s)$.

Proof. By the central limit theorem, the sequence $d\nu^{(n)}(t)$, the n-fold convolutions of $d\nu(\sqrt{n}\,t)$ with itself, converges to $(2\pi)^{-\frac{1}{2}} e^{-\frac{t^2}{2}}$ in the weak topology of $C^0(\mathbf{R})$, and furthermore, the moments of $d\nu^{(n)}(t)$ will converge to the moments of $e^{-\frac{t^2}{2}}(2\pi)^{-\frac{1}{2}}$. That is,

$$\int_{\mathbf{R}} f(t)|t|^m d\nu^{(n)}(t) \to \int_{\mathbf{R}} f(t)|t|^m \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \tag{6.18}$$

for every $m > -1$ and $f \in C^0(\mathbf{R})$. Thus

$$\int_{\mathbf{R}} f(t)|t|^m d\nu^{(n)}(t)$$

$$\int_{\mathbf{R}^n} f(t_1 + \cdots + t_n)|t_1 + \cdots + t_n|^m d\nu_1\left(t_1\sqrt{n}\right) \cdots d\nu_1\left(t_1\sqrt{n}\right)$$

$$= \int_{\mathbf{R}^n} f(\sigma(\overline{t}))|\sigma(\overline{t})|^m d\nu_n(\overline{t}) \tag{6.19}$$

Observe that the integral in (6.19) equals to

$$\sum f\left(\pm\frac{1}{\sqrt{n}} \pm \ldots \pm \frac{1}{\sqrt{n}}\right)\left|\pm\frac{1}{\sqrt{n}} \pm \ldots \pm \frac{1}{\sqrt{n}}\right|^m,$$

where the sum is taken over all possible combinations of n signs, (and thus the sum has 2^n terms).

From (6.19) follows that

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}|T^\alpha_\omega k(\sigma(\overline{s}))|^q\,|\sigma(\overline{s})|^{2\gamma+1}\frac{d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\gamma}=\int_{\mathbf{R}}|T^\alpha_\omega k(s)|^q\,\frac{e^{-\frac{s^2}{2}}|s|^{2\gamma+1}ds}{c_\gamma}$$

and

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}|k(\sigma(\overline{s}))|^p\,|\sigma(\overline{s})|^{2\beta+1}\frac{d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\beta}=\int_{\mathbf{R}}|k(s)|^p\,\frac{e^{-\frac{s^2}{2}}|s|^{2\beta+1}ds}{c_\beta}.$$

Therefore,

$$\lim_{n\to\infty}\frac{\left(\int_{\mathbf{R}^n}|T^\alpha_\omega k(\sigma(\overline{s}))|^q\,|\sigma(\overline{s})|^{2\gamma+1}\frac{d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\gamma}\right)^{\frac{1}{q}}}{\left(\int_{\mathbf{R}^n}|k(\sigma(\overline{s}))|^p\,|\sigma(\overline{s})|^{2\beta+1}\frac{d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\beta}\right)^{\frac{1}{p}}}$$

$$=\frac{\left(\int_{\mathbf{R}}|T^\alpha_\omega k(s)|^q\,\frac{e^{-\frac{s^2}{2}}|s|^{2\gamma+1}}{c_\gamma}ds\right)^{\frac{1}{q}}}{\left(\int_{\mathbf{R}}|k(s)|^p\frac{|s|^{2\beta+1}}{c_\beta}ds\right)^{\frac{1}{p}}}. \tag{6.20}$$

Recall that $k(s)$ is an even polynomial, and hence a linear combination of polynomials of the form of $L^\alpha_k(s^2)$, where L^α_k is a Laguerre polynomials.

We show that $L^\alpha_m\left(\sigma(\overline{s})^2\right)$ can be $d\nu_n(\overline{s})$ approximated with a linear combination of homogeneous functions of degree $2m$.

We will need the following Lemma, whose proof will be postponed to the Appendix 3.

Lemma 6.3 *For every* $\alpha\geq-\frac{1}{2}$, $m\geq 0$, $n\geq 1$, *and* $\overline{s}=\left(\pm\frac{1}{\sqrt{n}},\cdots,\pm\frac{1}{\sqrt{n}}\right)$,

$$L^\alpha_m(\sigma^2(\overline{s}))=\Phi^\alpha_m(\overline{s})+\frac{1}{n}\mathcal{R}^\alpha_m(\sigma(\overline{s})),$$

where $\mathcal{R}^\alpha_m(x)$ *is a polynomial of degree* $\leq 2m-2$ *whose coefficients depend only on* m *and* α, *and* $\Phi^\alpha_m(\overline{s})$ *is a homogeneous function of degree* $2m$ *which* *is defined as follows.*

$$\Phi_m^\alpha(\overline{s}) = \begin{cases} \dfrac{(-1)^m}{2^{2m}m!}\psi_{2m,\,n}(\overline{s}) & \textit{if } \alpha = -\frac{1}{2}, \\ \displaystyle\sum_{j=0}^{2m} \eta_{m,\,j}^\alpha \sigma(\overline{s})^j \psi_{2m-j,\,n}(\overline{s}) & \textit{if } \alpha \geq -\frac{1}{2}, \end{cases} \tag{6.21}$$

with

$$\eta_{m,\,j}^\alpha = \frac{(-1)^m \Gamma(m+\alpha+1)}{\pi^{\frac{1}{2}}(2m)!\Gamma(\alpha+\frac{1}{2})} 2^j \binom{2m}{j} \int_{-1}^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} (t-1)^j \, dt. \tag{6.22}$$

Furthermore, for every $l > -1$ and every $q \geq 1$,

$$\lim_{n\to\infty} \left(\int_{\mathbf{R}^n} \left| \Phi_m^\alpha(\overline{s}) - L_m^\alpha(\sigma^2(\overline{s})) \right|^q |\sigma(\overline{s})|^{2l+1} d\nu_n(\overline{s}) \right)^{\frac{1}{q}} = 0. \tag{6.23}$$

Next, we shall define an operator on $\mathbf{X_n}$ that approximates T_ω^α. Recall that

$$T_\omega^\alpha \left(\sum_{k=0}^M c_k L_k^\alpha(s^2) \right) = \sum_{k=0}^M c_k \omega^k L_k^\alpha \left(s^2\right).$$

Since $L_m^\alpha\left(\sigma(\overline{s})^2\right)$ can be approximated, in the sense of the previous Lemma, with the $\Phi_j^\alpha(\overline{s})$'s, the natural replacement for T_ω^α is the operator $\mathcal{K}_\omega = \mathcal{K}_{\omega,\,n} :$ $\mathbf{X_n} \to \mathbf{X_n}$

$$\mathcal{K}_\omega \left(\sum_{k=0}^M c_k \phi_j(\overline{s}) \right) = \sum_{k=0}^M c_j \omega^j \phi_j(\overline{s}),$$

where the ϕ_j's are homogeneous generators of $\mathbf{X_n}$ of degree $2j$. Thus, $\mathcal{K}_\omega \phi_j(\overline{s}) = \omega^j \phi_j(\overline{s}) = \phi_j(\overline{s}\sqrt{\omega})$, and if we let $k(s) = \displaystyle\sum_{k=0}^M c_k L_k^\alpha(s^2)$, and $g(\overline{s}) = \sum c_j \Phi_j^\alpha(\overline{s})$, we obtain $\mathcal{K}_\omega g(\overline{s}) = g(s\sqrt{\omega})$.

By Lemma 6.3,

$$\lim_{n\to\infty} \left(\int_{\mathbf{R}^n} |k(\sigma(s)) - g(\overline{s})|^p \frac{|\sigma(\overline{s})|^{2\beta+1}}{(2\pi)^{-\frac{1}{2}} c_\beta} d\nu_n(\overline{s}) \right)^{\frac{1}{p}} = 0,$$

and

$$\lim_{n\to\infty}\left(\int_{\mathbf{R}^n}|T^\alpha k(\sigma(s)) - \mathcal{K}_\omega(g)(\overline{s})|^q \frac{|\sigma(\overline{s})|^{2\gamma+1}}{(2\pi)^{-\frac{1}{2}}c_\gamma} d\nu_n(\overline{s})\right)^{\frac{1}{q}} = 0.$$

Consequently,

$$\lim_{n\to\infty}\frac{\left(\int_{\mathbf{R}^n}|\mathcal{K}_\omega g(\overline{s})|^q \frac{|\sigma(s)|^{2\gamma+1}d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\gamma}\right)^{\frac{1}{q}}}{\left(\int_{\mathbf{R}^n}|g(\overline{s})|^p \frac{|\sigma(s)|^{2\beta+1}d\nu_n(\overline{s})}{(2\pi)^{-\frac{1}{2}}c_\beta}\right)^{\frac{1}{p}}} = \frac{\left(\int_{\mathbf{R}}|T_\omega^\alpha k(s)|^q \frac{|s|^{2\gamma+1}e^{-\frac{s^2}{2}}ds}{c_\gamma}\right)^{\frac{1}{q}}}{\left(\int_{\mathbf{R}}|k(s)|^p \frac{|s|^{2\beta+1}e^{-\frac{s^2}{2}}ds}{c_\beta}\right)^{\frac{1}{p}}} \tag{6.24}$$

and if we prove that, for every $n > 1$, the ratio on the left-hand side of (6.24) is < 1, then the ratio on the left-hand side of (6.24) is ≤ 1 as well.

Since we have observed that $\mathcal{K}_\omega g(\overline{s}) = g(\sqrt{\omega}\,\overline{s})$, we have proved the Lemma.

6.4 End of the proof of Theorem 1.2

Replacing T_ω^α with $\mathcal{K}_\omega$ is one of the most crucial steps of the proof because it allows to reduce the proof of the inequality (6.17) to the proof of the discrete inequality

$$\left(\int_{\mathbf{R}^n}|g(\overline{s}\sqrt{\omega})|^q \frac{|\sigma(\overline{s})|^{2\gamma+1}}{(2\pi)^{-\frac{1}{2}}c_\gamma} d\nu_n(\overline{s})\right)^{\frac{1}{q}}$$
$$\leq \left(\int_{\mathbf{R}^n}|g(\overline{s})|^p \frac{|\sigma(\overline{s})|^{2\beta+1}}{(2\pi)^{-\frac{1}{2}}c_\beta} d\nu_n(\overline{s}))\right)^{\frac{1}{p}},\ g \in \mathbf{X_n}, \tag{6.25}$$

(see (6.16)), when n is sufficiently large. Recall that $1 < p \leq q < \infty$, $-\frac{1}{q} < \frac{2\gamma+1}{q} \leq \frac{2\beta+1}{p} + \frac{1}{p'} - \frac{1}{q}$, $c_m = \int_{\mathbf{R}} e^{-\frac{x^2}{2}}|x|^{2m+1}dx = \Gamma(m+1)2^{m+1}$, $d\nu_n(\overline{s}) = d\nu(\overline{s}\sqrt{n}) = d\nu(x_1\sqrt{n})\cdots d\nu(x_n\sqrt{n})$, and $\sigma(\overline{s}) = x_1 + ... + x_n$.

When $\beta = \gamma = -\frac{1}{2}$, $q = p'$ and $\omega = \overline{\omega} = p(p')^{-1}$, (6.25) has been proved by Beckner in [Be]. That is, Beckner proved the following unweighted inequality:

$$\left(\int_{\mathbf{R}^n}\left|g(\overline{s}\sqrt{\overline{\omega}})\right|^q d\nu_n(\overline{s})\right)^{\frac{1}{q}} \leq \left(\int_{\mathbf{R}^n}|g(\overline{s})|^p d\nu_n(\overline{s}))\right)^{\frac{1}{p}},\ g \in \mathbf{X_n}. \tag{6.26}$$

Beckner proved (6.26) for $q = p'$ with iterated applications of the following "two-point inequality".

$$\left(\frac{|A\sqrt{\overline{\omega}} + B|^{p'} + |A\sqrt{\overline{\omega}} - B|^{p'}}{2}\right)^{\frac{1}{p'}} \leq \left(\frac{|A + B|^p + |A - B|^p}{2}\right)^{\frac{1}{p}}. \quad (6.27)$$

The weighted inequality (6.25) cannot be proved in the same manner and its proof seem quite difficult.

We prove (6.25) for $q = p'$, $\omega = \overline{\omega} = p(p')^{-1}$ and $\dfrac{2\beta + 1}{p} = \dfrac{2\gamma + 1}{p'}$. That will conclude the proof of Theorem 1.2 since this is the case that correspond to $\nu = \frac{1}{2}$ and $\mu = 0$.

We argue by induction on n. When $n = 1$, $\overline{s} = s = \pm 1$, and $\sigma(s)$ takes only the values ± 1. (6.17) is equivalent to

$$\left(\int_{\mathbf{R}} |g(\sqrt{\overline{\omega}}\, s)|^{p'} d\nu_n(s)\right)^{\frac{1}{p'}} \leq (2\pi)^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p'})} (c^{\beta})^{-\frac{1}{p}} (c^{\gamma})^{\frac{1}{p'}} \left(\int_{\mathbf{R}} |g(s)|^p d\nu_n(\overline{s})\right)^{\frac{1}{p}}.$$

By (6.26) the following inequality holds true:

$$\left(\int_{\mathbf{R}} |g(\sqrt{\overline{\omega}}\, s)|^{p'} d\nu_n(s)\right)^{\frac{1}{p'}} \leq \left(\int_{\mathbf{R}} |g(s)|^p d\nu_n(s)\right)^{\frac{1}{p}}.$$

So, if $1 \leq (2\pi)^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p'})} (c^{\gamma})^{\frac{1}{p'}} (c^{\beta})^{-\frac{1}{p}}$, or equivalently, if

$$(c^{\beta})^{\frac{1}{p}} \leq (2\pi)^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p'})} (c^{\gamma})^{\frac{1}{p'}}, \quad (6.28)$$

then (6.17) follows. When $\frac{2\beta+1}{p} = \frac{2\gamma+1}{p'}$, by Hölder's inequality

$$c_{\beta}^{\frac{1}{p}} = \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{\frac{2\beta+1}{p}} dx\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{\frac{p'}{p}(2\beta+1)} dx\right)^{\frac{1}{p'}} \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} dx\right)^{\frac{1}{p} - \frac{1}{p'}}.$$

Since $\frac{2\beta+1}{p} = \frac{2\gamma+1}{p'}$ and $\int_{\mathbf{R}} e^{-\frac{x^2}{2}} dx = (2\pi)^{-\frac{1}{2}}$, then,

$$c_{\beta}^{\frac{1}{p}} \leq (2\pi)^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p'})} \left(\int_{\mathbf{R}} e^{-\frac{x^2}{2}} |x|^{2\gamma+1} dx\right)^{\frac{1}{p'}} = (2\pi)^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p'})} c_{\gamma}^{\frac{1}{p'}},$$

and (6.28) follows.

We now assume that (6.17) is valid for $n \geq 1$ and we prove that the same is true for $n+1$. We let $\overline{s} = (\overline{s}',\, s_{n+1})$, with $\overline{s}' \in \mathbf{R}^n$, and $d\nu_1(\overline{s}) = d\nu_1(\overline{s}')d\nu_1(s_{n+1})$. We also let

$$\tilde{g}(\overline{s}) = \begin{cases} g(\overline{s}) \left(\dfrac{|\sigma(\overline{s})|}{|\sigma(\overline{s}')|}\right)^{\frac{2\gamma+1}{p'}} & \text{if } |\sigma(\overline{s}')| \neq 0, \\ 0 & \text{if } |\sigma(\overline{s}')| = 0 \end{cases}$$

With this notation,

$$\left(\int_{\mathbf{R}^{n+1}} \left| g(\sqrt{\varpi}\,\overline{s})\right|^{p'} \frac{|\sigma(\overline{s})|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s})\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}} \left|\tilde{g}(\sqrt{\varpi}\,\overline{s})\right|^{p'} d\nu_1(s_{n+1})\right) \frac{|\sigma(\overline{s}')|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s}')\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}} \left|\tilde{g}(\sqrt{\varpi}\,\overline{s}',\, \sqrt{\varpi}\, s_{n+1})\right|^{p'} d\nu_1(s_{n+1})\right) \frac{|\sigma(\overline{s}')|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s}')\right)^{\frac{1}{p'}},$$

and by (6.26),

$$\leq \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}} \left|\tilde{g}(\sqrt{\varpi}\,\overline{s}',\, s_{n+1})\right|^{p} d\nu_1(s_{n+1})\right)^{\frac{p}{p'}} \frac{|\sigma(\overline{s}')|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s}')\right)^{\frac{1}{p'}}.$$

We recall the following convexity type inequality,

$$\left(\int_S \left(\int_T |f(s,t)|^p \mu(dt)\right)^{\frac{q}{p}} \nu(ds)\right)^{\frac{1}{q}} \leq \left(\int_T \left(\int_S |f(s,t)|^q \nu(ds)\right)^{\frac{p}{q}} \mu(dt)\right)^{\frac{1}{p}} \tag{6.29}$$

which holds for every positive measure spaces $(S,\, \mathcal{S},\, \nu)$, $(T,\, \mathcal{T},\, \mu)$, every measurable function $f(s,\, t)$ and every $0 < p \leq q < \infty$. By (6.29) and our initial assumptions,

$$\left(\int_{\mathbf{R}^{n+1}} \left| g(\sqrt{\varpi}\,\overline{s})\right|^{p'} \frac{|\sigma(\overline{s})|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s})\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^n} \left|\tilde{g}(\sqrt{\varpi}\,\overline{s}',\, s_{n+1})\right|^{p'} \frac{|\sigma(\overline{s}')|^{2\gamma+1}}{c^\gamma} d\nu_1(\overline{s}')\right)^{\frac{p}{p'}} d\nu_1(s_{n+1})\right)^{\frac{1}{p}},$$

$$\leq \left(\int_{\mathbf{R}} \int_{\mathbf{R}^n} |\tilde{g}(\overline{s})|^p \frac{|\sigma(\overline{s}')|^{2\beta+1}}{c^\beta} d\nu_1(\overline{s}') d\nu_1(s_{n+1}) \right)^{\frac{1}{p}},$$

and since $\tilde{g}(\overline{s}) = g(\overline{s}) \left(\frac{|\sigma(\overline{s})|}{|\sigma(\overline{s}')|} \right)^{\frac{2\beta+1}{p}}$ and $d\nu_1(\overline{s}')d\nu_1(s_{n+1}) = d\nu_1(\overline{s})$, we obtain (6.25) in the special case $q = p'$, $\omega = \overline{\omega} = p(p')^{-1}$ and $\dfrac{2\beta+1}{p} = \dfrac{2\gamma+1}{p'}$. The proof of Theorem 1.2 is thus concluded.

Appendix 1. A few counterexamples

We are left to show that range of value of ν in (4.2) is optimal.

- $\nu \leq \dfrac{1}{2}$ *is necessary.* We recall the identity

$$\int_0^1 (xy)^{\frac{1}{2}} J_\alpha(xy) x^{\alpha+\frac{1}{2}} (1-x^2)^s dx = 2^s \Gamma(s+1) y^{-s-\frac{1}{2}} J_{\alpha+s+1}(y) \quad (6.1)$$

which is valid for every $s > -1$. See [EMOT].
Take $\nu = \frac{1}{2} + 2\epsilon$, with $\epsilon > 0$, and $s = -\frac{1}{p} + \epsilon$, so that $f(x) = x^{\alpha+1-2\epsilon}(1-x^2)^{-\frac{1}{p}+\epsilon}\chi_{(0,1)}$, where $\chi_{(a,\,b)}(t)$ is the characteristic function of $(a,\,b)$, is in $L^p(0,\,1)$. Then,

$$\begin{aligned} L^\alpha_{\nu,\,\mu} f(y) &= y^\mu \int_0^1 (xy)^{\frac{1}{2}+\epsilon} J_\alpha(xy) x^{\alpha+\frac{1}{2}-2\epsilon} (1-x^2)^{-\frac{1}{p}+\epsilon} dx \\ &= y^{\mu+2\epsilon} \int_0^1 (xy)^{\frac{1}{2}} J_\alpha(xy) x^{\alpha+\frac{1}{2}} (1-x^2)^{-\frac{1}{p}+\epsilon} dx \\ &= y^{\mu+\epsilon+\frac{1}{p}-\frac{1}{2}} 2^{-\frac{1}{p}+\epsilon} \Gamma(1-\frac{1}{p}+\epsilon) J_{\alpha+\frac{1}{p'}+\epsilon}(y). \end{aligned}$$

Recalling that $\mu = \frac{1}{p'} - \frac{1}{q}$ and that $J_{\alpha+\frac{1}{p'}+\epsilon}(y) = \mathcal{O}(y^{-\frac{1}{2}})$ when $y \sim \infty$, we can see at once that $L^\alpha_{\nu,\,\mu} f(y) \sim y^{-\frac{1}{q}+\epsilon}$ when $y \sim \infty$, and hence does not belong to $L^q(\mathbf{R}^+)$.

- $\nu > -\alpha - \frac{1}{p'}$ *is necessary.* We now let $\nu = -\alpha - \frac{1}{p'} - \epsilon$, with $\epsilon > 0$. We let $F(x) = x^{1+\frac{1}{p'}+\epsilon+2\alpha}(1-x^2)^{-\frac{1}{p}+\epsilon}\chi_{(0,\,1)}(x)$. By the identity (6.1),

$$L^\alpha_{\nu,\,\mu} F(y) = y^{\nu-\frac{1}{2}+\mu} \int_0^1 (xy)^{\frac{1}{2}} x^{(\nu-\frac{1}{2})+1+\frac{1}{p'}+\epsilon+2\alpha} (1-t^2)^{-\frac{1}{p}+\epsilon} J_\alpha(xt) dt$$

$$= y^{-\alpha-\frac{1}{p'}-\epsilon-\frac{1}{2}+\mu} \int_0^1 (xy)^{\frac{1}{2}} x^{\alpha+\frac{1}{2}} (1-x^2)^{-\frac{1}{p}+\epsilon} J_\alpha(xt)dt$$
$$= 2^{-\frac{1}{p}+\epsilon}\Gamma(1-\frac{1}{p}+\epsilon) y^{-\alpha-\frac{1}{q}-\epsilon-\frac{1}{2}} J_{\alpha+\frac{1}{p'}+\epsilon}(y)$$

Recalling that $J_{\alpha+\frac{1}{p'}+\epsilon}(y) = \mathcal{O}(y^{\alpha+\frac{1}{p'}+\epsilon})$ when $y \sim 0$, we can see at once that $L^\alpha_{\nu,\mu}F(y) \sim y^{-\epsilon-\frac{1}{q}}$, and hence is not in $L^q(\mathbf{R}^+)$.

Appendix 2. Proof of Proposition 2.1

It is easy to see that Gaussian functions attain the equality in (2.10). By (2.2), the Fourier-Bessel transform of order $\alpha = \frac{n}{2} - 1$ is a constant multiple of the restriction of the Fourier transform to radial functions of $\mathbf{R}^n$. Consequently,

$$\|\widetilde{\mathcal{H}}_{\frac{n}{2}-1} f\|_{L^{p'}(\mathbf{R}^+,\ r^{n-1}dr)} = (2\pi)^{-\frac{n}{2}} |S^{n-1}|^{\frac{1}{p'}} \left(\int_0^\infty |\hat{f}(r)|^{p'} r^{n-1} dr\right)^{\frac{1}{p'}}$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}}}{(2\pi)^{\frac{n}{2}} 2^{\frac{1}{p'}} \pi^{\frac{n}{2p'}}} \left(\int_{\mathbf{R}^n} |\hat{f}(|x|)|^{p'} dx\right)^{\frac{1}{p'}}. \tag{6.2}$$

Furthermore,

$$||f||_{L^p(\mathbf{R}^+,\ r^{n-1}dr)} = \frac{\Gamma\left(\frac{n}{2}\right)^{\frac{1}{p}}}{2^{\frac{1}{p}} \pi^{\frac{n}{2p}}} \left(\int_{\mathbf{R}^n} |f(|x|)|^p dx\right)^{\frac{1}{p}}, \tag{6.3}$$

and from (6.3) and (6.2) and the Theorems of Beckner and Lieb follows that

$$\frac{||\widetilde{\mathcal{H}}_{\frac{n}{2}-1} f||_{L^{p'}(\mathbf{R}^+,\ r^{n-1}dr)}}{||f||_{L^p(\mathbf{R}^+,\ r^{n-1}dr)}} = \frac{2^{\frac{1}{p}-\frac{1}{p'}} \pi^{\frac{n}{2p}-\frac{n}{2p'}}}{(2\pi)^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^{p'} dx\right)^{\frac{1}{p'}}}{\left(\int_{\mathbf{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}}}$$
$$\leq \frac{2^{\frac{1}{p}-\frac{1}{p'}} \pi^{\frac{n}{2p}-\frac{n}{2p'}}}{(2\pi)^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} (2\pi)^{\frac{n}{p'}} (p^{\frac{1}{p}} (p')^{-\frac{1}{p'}})^n.$$
$$= \Gamma\left(\frac{n}{2}\right)^{\frac{1}{p'}-\frac{1}{p}} \frac{p^{\frac{n}{2p}}}{(p')^{\frac{n}{2p'}}} 2^{\frac{(n-2)(2-p')}{2p'}}$$

as required.

Appendix 3. Proof of Lemma 6.3

Let $H_m(x)$ be the classical Hermite polynomial of degree m. Beckner proved in [Be] that the functions $H_m(\sigma(\overline{s}))$ can be $d\nu_n(\overline{s})$-approximated by symmetric functions. That is, for every $\overline{s} = \left(\pm\frac{1}{\sqrt{n}}, \cdots, \pm\frac{1}{\sqrt{n}}\right)$,

$$H_m(\sigma(\overline{s})) = \psi_{m,n}(\overline{s}) + \frac{1}{n}\sum_{r=1}^{[\frac{m}{2}]} a_{m,r} H_{m-2r}(\sigma(\overline{s})) \tag{6.4}$$

where $\sigma(\overline{s}) = x_1 + ... + x_n$, and the $a_{m,r}$ are bounded with respect to n for a fixed m. We recall that $L_m^{-\frac{1}{2}}(\zeta^2) = \dfrac{(-1)^m}{2^{2m}m!}H_{2m}(\zeta)$. When $\alpha > -\frac{1}{2}$, the following identity holds

$$L_m^{\alpha}\left(\zeta^2\right) = \frac{(-1)^m}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\frac{\Gamma(m+\alpha+1)}{(2m)!}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}H_{2m}(\zeta t)\,dt, \tag{6.5}$$

(see e.g. [Sz]).

We prove the Lemma for $\alpha > -\frac{1}{2}$, since the proof is quite similar in the other case. The derivatives of H_k satisfy the following identity:

$$\frac{d^j}{d^j\zeta}H_k(\zeta) = 2^j j! \binom{k}{j} H_{k-j}(\zeta), \qquad j \leq k.$$

By Taylor's formula

$$H_k(\zeta t) = \sum_{j=0}^{k}(2\zeta)^j(t-1)^j\binom{k}{j}H_{k-j}(\zeta),$$

and by (6.5),

$$L_m^{\alpha}\left(\frac{\zeta^2}{2}\right)$$

$$= c_{m,\alpha}\sum_{j=0}^{2m}(2\zeta)^j\binom{2m}{j}H_{2m-j}(\zeta)\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}(t-1)^j\,dt, \tag{6.6}$$

where we lave let $c_{m,\alpha}$ be the constant on the right-hand side of (6.5). By (6.6) and (6.4) the conclusion follows.

To prove (6.23) we recall that the moments of $d\nu_n(\overline{s})$ converge to the moments of $\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$ in the weak topology of $C^0(\mathbf{R})$; thus,

$$\lim_{n\to\infty}\int_{\mathbf{R}^n}|\mathcal{R}_m^\alpha(\sigma(\overline{s}))|^q|\sigma(\overline{s})|^{2l+1}d\nu_n(\overline{s}) = \int_{\mathbf{R}}|\mathcal{R}_m^\alpha(x)|^q|x|^{2l+1}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx < \infty$$

and $\lim_{n\to\infty}\frac{1}{n^q}\int_{\mathbf{R}^n}|\mathcal{R}_m^\alpha(\sigma(\overline{s}))|^q|\sigma(\overline{s})|^{2l+1}d\nu_n(\overline{s}) = 0$.

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