Geometric properties of the zero sets of harmonic functions in the plane

I will present 2 joint papers with Steve Hudson from FIU and also some work in progress. Our joint work deals with the geometry of level sets of harmonic functions in the plane and with the unique continuation properties of these sets.

Level sets of harmonic functions have remarkable properties. For example, it is well known that if Z, the set where a harmonic function u vanishes, contains a simple closed curve, then u vanishes inside, by the maximum principle, and thus it vanishes on R^2.

In [F] the author shows that if a curve in Z is parametrized by $x = p(t)$ and $y = q(t)$, where $p(t)$ and $q(t)$ are polynomials with real coefficients, then the curve is a straight line or a parabola. Therefore the zero set of an harmonic function can contain $y = x^2$ but it cannot contain a cubic, such as $y = x^3$. See also

[FNS]. These results do not apply, for example, to $u(x,y) = xy+1$, since hyperbolas cannot be parametrized by polynomials.

It is not clear whether any simple geometric properties distinguish curves like $y = x^3$ from those like $y = x^2$. It is certainly possible for Z to have an inflection point. For example, let $u = x^3 - 3xy^2 + x + y$. Then, the origin is an inflection point of Z.

In our paper [LS1] (currently under review) Steve Hudson and I we tried to rule out other "implausible" zero sets, in particular the ones that "wiggle very much". That is, those sets whose curvature changes often.

Let me recall the stndard formulas for the curvature of the graph of a function $y = f(x) \in C^2(\mathbf{R})$ at a point $P = (p, f(p))$: it is

$$Kf(P) = \frac{f''(p)}{(1 + (Ff'(p))^2)^{\frac{3}{2}}}.$$

By the implicit function theorem, the zero set of u is locally graph of a C^2 function whenever

$\nabla u \neq 0$. Using the implicit function theorem we can deduce the following formula for the curvature.

$$Ku(P) =$$

$$-\frac{u_y^2(P)u_{xx}(P) - 2u_{xy}(P)u_x(P)u_y(P) + u_x^2(P)u_{yy}(P)}{\left(u_x^2(P) + u_y^2(P)\right)^{\frac{3}{2}}}$$

$Ku(P)$ is a function which associates to each $P \in \Omega - \{(x,y) : \nabla(x,y) = (0,0)\}$ the curvature of the level set of u that passes through P. We say in short that $Ku(P)$ is the curvature of u at P.

I would like to recall also the less known curvature formula.

If $u = \text{Re}(w)$, where w is an holomorphic function in Ω, then the curvature of a level set of u at point P where $\nabla u(P) \neq 0$ is

$$K(P) = |w'(P)|\text{Re}\left(\frac{w''(P)}{(w'(P))^2}\right). \qquad (1)$$

It seems very difficult to control the curvature of a level set of an harmonic function. For example, $u(x,y) = xy$ has undefined curvature at $(0,0)$ and a perturbation such as $u = xy - \epsilon^2$ leads to a very large curvature at (ϵ, ϵ), (that is, $Ku(\epsilon,\epsilon) \sim 1/\epsilon$).

However, if Z is a *single* connected curve in the unit ball, then κ is bounded at any point $P \in Z$. The bound depends only on the distance from P to the boundary but not on u. Our main result assumes for simplicity that P is the origin and says the following

Theorem 1 *Suppose that $\triangle u = 0$ on $B_r(0) = \{|x| \leq r\} \subset R^2$ with boundary values $f(\theta)$. Assume $f(\theta) \geq 0$ on an interval $I = [\alpha, \beta] \subset [-\pi, \pi]$ and $f(\theta) \leq 0$ otherwise. Suppose $u(0,0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)d\theta = 0$. Then Z is a curve and its curvature at (0,0) is bounded by $\frac{C}{r}$, where $C \leq 24$ does not depend on u.*

The proof of this Theorem is easy when $[\alpha,\ \beta] = [0,\ \pi]$; we though that the generalization to other intervals was no big deal, but we quickly realized that that was not the case; in fact the proof of the general case is rather hard and technical. If time allows I would like to present Steve's original proof rather than its subsequent generalization

Outline of Proof: Assume WLOG that $u_y(0,0) = 0$ and $u_x(0,0) = 1$ For such u, the curvature at (0,0) is u_{yy}. Also, $u = f * K$, where $K = (1 - r^2)/(1 - 2r\cos(\theta) + r^2)$ is the Poisson kernel. At (0,0) we get $K_x = 2\cos(\theta)$ and $K_{yy} = -4\cos(2\theta) = -8\cos^2(\theta) + 4$

So, $Ku(0,0) = u_{yy} = f * K_{yy} = -8 \int f \cdot \cos^2(\theta)$ (since $\int f = 0$). But $f \cdot \cos(\theta) \geq 0$ and $\cos(\theta) \leq 1$, so $\kappa \geq -8 \int f \cdot \cos(\theta) = -8u_x = -8$.

Likewise, $\int f \cdot \cos^2(\theta) > -\int f \cdot \cos(\theta) = -u_x$, so $Ku(0,0) \leq 8$.

Theorem 1 has several consequences. The following Corollary shows how the curvature of Z at a grows with the distance of a from the boundary of the disk.

Corollary 2 *Let u be a harmonic function in the unit disk. Suppose that Z is a single connected curve, as in Theorem 1. Then, for every $a \in Z$, the curvature $\kappa(u)(a)$ of Z at a satisfies the following inequality.*

$$|\kappa u(a)| \leq \frac{C + 2|a|}{1 - |a|^2} \leq \frac{C_2}{1 - |a|}. \qquad (2)$$

where C is the constant in Theorem 1 and C_2 is some constant independent of both a and u.

A variation of the hyperbola example shows that the bound of $Ku(a)$ in terms of $(1-|a|)^{-1}$ is sharp.

If Z is a single curve in the entire plane that intersects any circle or radius r only two times (the case $r = \infty$ of Theorem 1), we might heuristically conclude that $\kappa = 0$ at every point of Z, so that Z and u must be linear. This claim is true, but we can prove only a special case of it directly from Theorem 1. We proved the general version of this Theorem using complex variables. Likewise, if Z consists of m curves, u must be a polynomial of degree m; this is a slightly improved version of a theorem in [WWZ].

Theorem 3 *Let u be harmonic on the plane, and assume $U_+ = \{z : u(z) > 0\}$ and $U_- = \{z : u(z) < 0\}$ are both connected. Then u is linear.*

Theorem 4 *If U_+ and U_- have a total of n components, then u is a polynomial of degree m, with $\frac{n}{2} \leq m \leq n-1$.*

Proof of Theorem 3: We present here an elegant complex analytic proof, found independently by Bao Qin Li. Let $f = v + iu$ be analytic on the plane. The zeroes of f all lie in Z. Suppose n zeroes lie inside a simple closed curve Γ, which intersects Z just twice. So, $f(\Gamma)$ winds around the origin at most once. By the Argument Principle, $n \leq 1$. So, f has at most one zero in the plane. The same is true for $f - a$, for any $a \in \mathbf{R}$. So, f must be a polynomial (otherwise, $f(z) = a$ would occur infinitely often, for most values of a). Since f has only one zero (counting multiplicity), it is linear.

We also proved that when Z contains a line segment, it contains a full line.

The following theorem can be viewed as a unique continuation theorem for curvature.

Theorem 5 *a) Let u be analytic in Ω. Suppose that $|\nabla u| \neq 0$ in Ω. If the points where the curvature of u vanishes lie on a straight line l and have at least a limit point, then the curvature of u vanishes identically on l.*

b) If u is an harmonic polynomial on $\mathbf{R}^2$, then it is the sum of a linear function and an harmonic polynomial that vanishes on l.

Our proof works only for polynomials and does not seem easily generalizable to general harmonic functions, but we conjecture that the theorem is valid also in general.

Several of these theorems can be viewed as uniqueness results for harmonic functions, based on properties of Z. They partially answer a very general problem; suppose that u is harmonic on a domain $D \subseteq R^n$ and $u \equiv 0$ on a set $Z \subseteq D$. What properties of Z imply that $u \equiv 0$ on all of D? For example, if Z contains an open ball, then u vanishes on D; this is often called the *unique continuation property* (UCP). Or, if Z is a single point, and u vanishes with all its derivatives there (we also say that u vanishes to infinite order at this point) then u vanishes on D; this is the *strong unique continuation property* (SUCP). These properties extend to certain classes of nonharmonic functions Suppose

$$\triangle u(x) \leq V(x)|u(x)|, \tag{3}$$

where V is called the potential of u. If $V \in L^{\frac{n}{2}}_{\text{loc}}(D)$ (where $D \subseteq R^n$), then u has the UCP and SUCP * There is an extensive literature

*In this context, that $u \in W^{2,p}(D)$ vanishes to infinite order at x_0 if for every $N > 0$,

$$\lim_{\epsilon \to 0} \epsilon^{-N} \int_{|x-x_0|<\epsilon} |u(x)|^2 dx = 0.$$

on these problems; See for example the survey paper of T. Wolff [Wo] and a recent article of H. Koch and D. Tataru [KT] and the literature cited there.

Typically, unique continuation proofs depend on Carleman estimates. Steve and I attempted a new approach in [DH], and became curious about the intermediate case, in which Z is a set of positive dimension $0 < d < n$, rather than an open ball, or a single point.

In [LS2] we have proved that any harmonic function u on $\mathbf{R}^2$ that vanishes on a piece of an analytic curve Γ must vanish on the whole curve. If Γ is a line, then u must be linear.

None of these results is valid for non harmonic solutions of (3). We prove the following

Theorem 6 *There is a function $u : R^2 \to R$, with $V \in L^\infty$, that vanishes on a ray in R^2, but not on the whole line. $||V||_\infty$ can be arbitrarily small.*

Any harmonic function in $\mathbf{R}^2$ that vanishes on a closed curve must vanish also inside, by the maximum principle, but for general solutions of (3), there is no maximum principle, and Z may contain closed curves. However, we prove that the area of a region enclosed by Z cannot be arbitrarily small. We prove a lower bound that depends on $||V||_\infty$.

Theorem 7 *Let $u(x) \in W_0^{2,p}(D)$, with D a domain of $\mathbf{R}^n$, be a nontrivial solution of the differential inequality (3). Assume $u(x) > 0$ in D and $u \equiv 0$ on the boundary of D.*

i) If $n = 2$ and $|V(x)|$ is bounded in D, then

$$|D| \geq 4\pi ||V||_\infty^{-1}. \tag{4}$$

ii) If $n > 2$ and $V \in L^r(D)$ with $r > \frac{n}{2}$, then

$$|D| \geq \left(\frac{n-2}{2||V||_r^{\frac{1}{2}}} \right)^{\frac{2nr}{2r-n}}. \tag{5}$$

iii) If $V \in L^{\frac{n}{2}}(D)$, with $n > 2$, then necessarily

$$||V||_{\frac{n}{2}} \geq \frac{(n-2)^2}{4}. \qquad (6)$$

The argument used to prove Theorem 7 can also be used to prove a unique continuation theorem. In fact, (6) shows that if $\frac{2}{n-2}||V||_{\frac{n}{2}}^{\frac{1}{2}} < 1$, then the only solution of the differential inequality (3) is $\equiv 0$.

The constants in Theorem 7 are not sharp. In our paper [LS3] (Still work in progress) we prove the following sharp version of Theorem 7 when $n = 2$.

Theorem 8 *Under the assumptions of Theorem 7,*

$$||V||_{\infty}|D| \geq \pi j_0^2 \qquad (7)$$

where $j_0 \approx 2.4048$ is the first zero of $J_0(x)$, the Bessel function of order zero. This value is attained by $U = J_0(r)$ and by $V = \Delta U/U$.

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