Mathematical Economics Midterm #2, November 6, 2003

1. Are the following sets open in $\mathbb{R}^2$? Closed? Neither? Explain.

   a) $A = \{(x, y) : x^2 + y^2 = 1\}$.

   Let $f(x, y) = x^2+y^2$. Because f is a quadratic polynomial, it is continuous. Now $A = f^{-1}(\{1\})$ and so A is closed because it is the inverse image of a closed set. However, A is not open. To see this consider $B_\varepsilon(1, 0)$. This contains points such as $(1, \varepsilon/2)$ that are not in the set A. Since no ball about $(1, 0) \in A$ is contained in A, A is not open.

   b) $B = \{(x, y) : x^2 + 2xy + 12y^2 \leq 36\}$.

   Let $f(x, y) = x^2+2xy+12y^2$. Because f is a quadratic polynomial, it is continuous. Now $B = f^{-1}((-\infty, 36])$ and so B is closed because it is the inverse image of a closed set. However, B is not open. To see this consider $B_\varepsilon(6, 0)$. This contains points such as $(6, \varepsilon/2)$ that are not in the set B. Since no ball about $(6, 0) \in B$ is contained in B, B is not open.

2. Consider the sequence $x_n = (-1)^n + n^2/(n^3 + 2n + 1)$.

   a) Does it converge? If so, what is its limit?

   It does not converge. Note that $|x_n - x_{n+1}| \to 2$.

   b) If it doesn't converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   Yes, it has convergent subsequences. Two obvious ones are the subsequence of even terms (with limit 1) and the subsequence of odd terms (with limit -1).

3. Consider the differential system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x + 3y \end{pmatrix}$$

   with initial conditions $x(0) = 2$ and $y(0) = 3$.

   a) Write the system in matrix form.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

   b) Find the eigenvalues of this system.

The eigenvalue equation is $0 = (3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8$. This has solutions $\lambda = 2$ and $\lambda = 4$.

c) Is this system stable?

No, both eigenvalues are positive so it is unstable.

d) Find the solution to this system.

Since the original matrix is symmetric, we can find orthonormal eigenvalues. One eigenvector corresponding to $\lambda = 4$ is $(1,1)'$ and an eigenvector corresponding to $\lambda = 2$ is $(1,-1)'$. In both cases we can normalize by dividing by $\sqrt{2}$ to obtain orthonormal eigenvectors.

This yields basis matrix:

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{with} \qquad B^{-1} = B' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

and then

$$B^{-1}AB = \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

The solution is

$$B^{-1}e^{\Lambda t}B\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5e^{4t} - e^{2t} \\ 5e^{4t} + e^{2t} \end{pmatrix}.$$

4. Robinson Crusoe has utility function $u(x,y) = \ln x + xy + y^2$. Crusoe has a production possibilities set given by $\{(x,y) : x \geq 1, y \geq 0, x + 2y \leq 4, 3x + y \leq 9\}$.

a) Is the production possibility set a closed set? Is it a bounded set? Explain.

Yes, the production possibility set is closed. It can be written as $\{(x,y) : x \geq 1\} \cap \{(x,y) : y \geq 0\} \cap \{(x,y) : x + 2y \leq 4\} \cap \{(x,y) : 3x + y \leq 9\}$. Each of these 4 sets is the inverse image of a closed set under a linear function and is closed. Thus the intersection is also closed.

The production possibilities set is also bounded. To see this, note that $1 \leq x \leq 3$ and $0 \leq y \leq 2$. It follows that $\|(x,y)\|^2 \leq 3^2 + 2^2 = 13$, so $\|(x,y)\| \leq \sqrt{13}$.

b) Show that Crusoe's problem of maximizing utility over his production set has a solution.

The production set is compact because it is closed and bounded. Because $x \geq 1$, the function $\ln x$ is continuous on the production set. The polynomial terms are also continuous, so the sum f is continuous. By the Weierstrass Theorem, this continuous function must have a maximum on the compact production possibilities set.

5. Let $f(x, y) = x^2/(x^2 + y^2)$ so that $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$.

   a) What is the range of f?

   Clearly $0 \leq f(x, y) \leq 1$. Both endpoints can occur: $f(0, 1) = 0$ and $f(1, 0) = 1$. The range is $[0, 1]$.

   b) Is f onto?

   No, $\operatorname{ran} f \neq \mathbb{R}^2$.

   c) Is f one-to-one?

   No, $f(0, 1) = f(0, 1/2) = 0$.

   d) Is f continuous?

   Yes. It is the quotient of continuous functions and the denominator is non-zero.

   e) Compute df

$$df = \left(\frac{2xy^2}{(x^2 + y^2)^2}, \frac{-2yx^2}{(x^2 + y^2)^2} \right).$$