Homework Assignment #7

16.1 Determine the definiteness of the following symmetric matrices:

$$a)\ \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad b)\ \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \quad c)\ \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix} \quad d)\ \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$e)\ \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \quad f)\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad g)\ \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

In each case, we start by computing the leading principal minors. If necessary, we will compute all principal minors.

a) The leading principal minors are $H_1 = 2 > 0$ and $H_2 = 1 > 0$, which implies the matrix is positive definite.

b) Here $H_1 = -3 < 0$ and $H_2 = -1 < 0$. This violates both sign patterns, which implies the matrix is indefinite.

c) Here $H_1 = -3 < 0$ and $H_2 = 2 > 0$, which implies the matrix is negative definite.

d) Now $H_1 = 2 > 0$ and $H_2 = 0$. Since $H_2 = 0$, we have insufficient information and must look at all principal minors. There is one other principal minor, $h_{22}8 > 0$. Since all principal minors are non-negative, the matrix is positive semidefinite.

e) Here $H_1 = 1 > 0$, $H_2 = 0$, and $H_3 = -25 < 0$. Since both sign patterns are violated by non-zero minors, the matrix is indefinite.

f) Now $H_1 = -1 < 0$, $H_2 = 0$, and $H_3 = 0$. Again we must examine all principal minors. The other first-order principal minors are -1 and -2, while the other second-order principal minors are both 2. Since all even order minors are non-negative, and all odd minors are non-positive, the matrix is negative semidefinite.

g) We compute $H_1 = 1 > 0$, $H_2 = 2 > 0$, $H_3 = -10 < 0$. Before even computing H_4, we see that both sign patterns are violated. The matrix is therefore indefinite.

16.6 Determine the definiteness of the following constrained quadratics:

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

c) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

d) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

e) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 + x_3 = 0$.

a) The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

There are $n = 2$ variables and $m = 1$ constraints, so we must compute the last leading principal minor $H_3 = 2$. Now $2(-1)^n > 0$, which implies the constrained quadractic is negative definite.

b) The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Again there are two variables and one constraint. Now $H_3 = -1$ and $(-1)(-1)^m > 0$, implying positive definite.

c) Now there are $n = 3$ variables and $m = 2$ constraints. The bordered Hessian is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{pmatrix}.$$

We compute H_5 by a combination of row-reduction and expansion.

$$H_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 2 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 16.$$

Since $H_5(-1)^m > 0$, the constrained quadratic is positive definite.

d) As before, $n = 3$ and $m = 2$. We compute:

$$H_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 2 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 16.$$

Since $H_5(-1)^m > 0$, the constrained quadratic is positive definite.

e) The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & -3 \\ 1 & 0 & -3 & -1 \end{pmatrix}.$$

Now $n = 3$ and $m = 1$, so we look at the last 2 leading principal minors. We easily compute $H_3 = 3$, while some calculations show $H_4 = 4$. But $(-1)^m H_4 = (-1)^n H_4 = -4 < 0$, which implies the constrained quadratic is indefinite

17.1 For each of the following functions defined on $\mathbb{R}^2$, find the critical points and classify these as local max, local min, saddle point, or "can't tell":

$$a)\ xy^2 + x^3y - xy, \qquad b)\ x^2 - xy + 2y^2 + 10x - 5,$$
$$c)\ x^4 + x^2 - 6xy + 3y^2, \qquad d)\ 3x^4 + 3x^2y - y^3.$$

a) The first-order conditions are:

$$0 = y^2 + 3x^2y - y$$
$$0 = 2xy + x^3 - x.$$

If $xy \neq 0$, we can divide by x and y to obtain

$$0 = y + 3x^2 - 1$$
$$0 = 2y + x^2 - 1.$$

These are easily solved to find $x = \pm 1/\sqrt{5}$ and $y = 2/5$. If $x = 0$, the first equation becomes $y^2 - y = 0$, with solutions $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$, while if $y = 0$, the second equation becomes $x^3 - x = 0$, yielding additional critical points $(\pm 1, 0)$. Thus we have six critical points.

The Hessian is
$$H = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

We check the leading principal minors. The key minors are $H_2(0, 1) = -1 < 0$, indefinite; $H_2(\pm 1, 0) = -4 < 0$, indefinite; $H_2(0, 0) = -1 < 0$, indefinite; $H_1(\pm 1/\sqrt{5}, 2/5) = \pm 6/\sqrt{5}$ and $H_2(\pm 1/\sqrt{5}, 2/5) = 2/5 > 0$, which is positive or negative definite depending on the sign of x. Thus $(0, 0)$, $(0, 1)$, and $(\pm 1, 0)$ are all saddlepoints, $(1/\sqrt{5}, 2/5)$ is a local minimum, and $(-1/\sqrt{5}, 2/5)$ is a local maximum.

b) The first-order conditions are:

$$0 = 2x - y + 10$$
$$0 = -x + 4y.$$

There is only one solution, $(-40/7, -10/7)$ and the Hessian is

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}.$$

Then $H_1 = 2 > 0$ and $H_2 = 7 > 0$ implies H is positive definite, so we have a local minimum.

c) The first-order conditions are:

$$0 = 4x^3 + 2x - 6y$$
$$0 = -6x + 6y.$$

The second equation tells us $x = y$. We substitute in the first equation to find $4x^3 - 4x = 0$. The critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$. The Hessian is

$$\begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

Since $H_2(0, 0) = -24 < 0$, $(0, 0)$ is a saddlepoint. Also, $H_1(1, 1) = H_1(-1, -1) = 14 > 0$ and $H_2(1, 1) = H_2(-1, -1) = 48 > 0$, so $(1, 1)$ and $(-1, -1)$ are local minima.

d) The first-order conditions are:

$$0 = 12x^3 + 6xy$$
$$0 = 3x^2 - 3y^2.$$

The second equation implies either $x = y$ or $x = -y$. Successively substituting in the first equation, we find critical points $(0, 0)$, $(1/2, -1/2)$, and $(-1/2, -1/2)$. The Hessian is

$$\begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}.$$

Then $H(0, 0)$ is the zero matrix, so we can't tell anything more about the critical point $(0, 0)$. [In fact, and $f(0, \epsilon) = -\epsilon^3$, which can have either sign, so $(0, 0)$ is actually a saddlepoint.] Also, $H_1(\pm 1/2, -1/2) = 6 > 0$ and $H_2(\pm 1/2, -1/2) = -27 < 0$, so the other two critical points are saddlepoints.

17.4 A firm uses two inputs to produce a single product. If its production function is $Q = x^{1/4}y^{1/4}$ and if it sells its output for a dollar a unit and buys each input for \$4 per unit, find its profit-maximizing input bundle. (Check the second-order conditions.)

Profit is $x^{1/4}y^{1/4} - 4x - 4y$. The first-order conditions are $(1/4)x^{-3/4}y^{1/4} = 4$ and $(1/4)x^{1/4}y^{-3/4} = 4$. The critical point is $(1/256, 1/256)$. The Hessian is:

$$\begin{bmatrix} \frac{-3}{16}x^{-7/4}y^{1/4} & \frac{1}{16}x^{-3/4}y^{-3/4} \\ \frac{1}{16}x^{-3/4}y^{-3/4} & \frac{-3}{16}x^{1/4}y^{-7/4} \end{bmatrix}$$

The first leading principal minor is negative, and the determinant of the Hessian is $(1/32)x^{-3/2}y^{-3/2} > 0$. The second-order conditions are satisfied, so we have a maximum.

17.5 More generally, suppose that a firm has a Cobb-Douglas production function $Q = x^a y^b$ and that it faces output price p and input prices w and r, respectively. Solve the first-order conditions for a profit-maximizing input bundle. Use the second-order conditions to determine the values of the parameters a, b, p, w, and r for which this solution is a global max.

Profit is $px^a y^b - wx - ry$. The first-order conditions are $apx^{a-1}y^b = w$ and $bpx^a y^{b-1} = r$. We will make the natural assumption that all prices are strictly positive. It follows that $a, b > 0$. We will require that the Hessian be negative definite, which implies $a + b < 1$.

The first FOC implies $y^b = (w/ap)x^{1-a}$, so $y = (w/ap)^{1/b}x^{(1-a)/b}$. Substituting in the other first order condition, we obtain $r = bpx^a(w/ap)^{(b-1)/b}x^{(b-1)(1-a)/b} = bp(w/ap)^{(b-1)/b}x^{a+b-1}$. It follows that

$$x = \left[\left(\frac{r}{bp}\right)^b \left(\frac{w}{ap}\right)^{1-b}\right]^{\frac{1}{a+b-1}} \quad \text{and} \quad y = \left[\left(\frac{r}{bp}\right)^{1-a} \left(\frac{w}{ap}\right)^a\right]^{\frac{1}{a+b-1}}$$

17.7 For the discriminating monopolist of Example 17.3, compute the demand function for the market as a whole, without price discrimination. Compute the firm's profit-maximizing output for this situation and compare the profit to the computation in Example 17.3.

The price is the same (p) in both markets. Quantity demanded obeys $p = 50 - 5Q_1$ in market 1 and $p = 100 - 10Q_2$ in market 2. Thus $Q_1 = 10 - p/5$ and $Q_2 = 10 - p/10$. Total quantity demanded is $Q = Q_1 + Q_2 = 20 - 3p/10$. Profits are then $p(Q_1 + Q_2) - (90 + 20(Q_1 + Q_2)) = 20p - 3p^2/10 - 90 - 400 + 6p = -490 + 26p - 3p^2/10$. The first-order conditions are $26 - 6p/10 = 0$, or $p = 130/3$. Notice that the second derivative is negative, so this is a maximum. Then $Q_1 = 4/3$ and $Q_2 = 17/3$, so $Q = 7$. It follows that profit is 220/3. Since total production is the same as in the example, the difference in profit is due to the difference in total revenue. Under the single-price regime, profit is $7(130/3) = 910/3 = 303\frac{1}{3}$. Under the two-price regime, revenue in market one is $35(3) = 105$, and revenue in market two is $60(4) = 240$, for a total of 345. Thus profit is $42\frac{1}{3}$ higher under the two-price regime.