Homework Assignment #6

14.4 Consider the production function $Q = 9L^{2/3}K^{1/3}$.

a) What is the output when $L = 1000$ and $K = 216$?

$9 \cdot 100 \cdot 6 = 5400$.

b) Use marginal analysis to estimate $Q(998, 216)$ and $Q(1000, 217.5)$.

Here $Q(998, 216)$ and $Q(1000, 217.5)$ can be approximated by $Q(1000, 216)+(\partial Q/\partial L)(1000, 216)(998-1000) = 5400 - 7.2 = 5392.8$ and $Q(1000, 216) + (\partial Q/\partial K)(1000, 216)(217.5) = 5400 + 12.5 = 5412.5$.

c) Use a calculator to compute these two values of Q to three decimal places and compare these values with your estimates in *b*.

$Q(998, 216) \approx 5392.798$ and $Q(1000, 217.5) \approx 5412.471$. The values are very close, differing by only 0.002 and 0.029, respectively.

d) How big must ΔL be in order for the difference between $Q(1000 + \Delta L, 216)$ and its linear approximation, $Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L$, to differ by more than two units?

The approximation is $5400 + 3.6\Delta L$. It differs by 2 units from the actual value when $\Delta L \approx 58.5$. How did I find this number? I knew from the previous parts that small ΔL would not work, so I started with $\Delta L = 100$. I calculated the error and discovered it was almost 6. Linear interpolation would suggest trying 33, but considering the function, I decided that $\Delta L = 50$ would be a good second try. It led to an error of -1.5. The next try was $\Delta L = 60$, with error 2.1. Linear interpolation gave $\Delta L = 58$ as the next try, with error 1.97. Another round of linear interpolation gave 58.5 with error 2.0017. I decided that was close enough to 2.0 to stop.

14.14 Calculate the rate of change of output with resepct to changes in r in Example 14.8 when $t = 10$ and $r = 0.1$.

First note that

$$\begin{aligned}\frac{\partial Q}{\partial r} &= \frac{\partial Q}{\partial K} \cdot \frac{\partial K}{\partial r} + \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial r} \\ &= (3K^{-1/4}L^{1/4}) \cdot (-10t^2r^{-2}) + (K^{3/4}L^{-3/4}) \cdot (250).\end{aligned}$$

Using $K(10, 0.1) = 10,000$, $L(10, 0.1) = 625$, $t = 10$, and $r = 0.1$, we obtain

$$(3 \cdot 10,000^{-1/4}625^{1/4}) \cdot (-10 \cdot 100 \cdot 100) + (10000^{3/4} \cdot 625^{-3/4}) \cdot (250) = -148000.$$

15.1

a) Prove that the expression $x^2 - xy^3 + y^5 = 17$ defines an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$.

We first note that $(5, 2)$ solves the equation. We then rewrite it as $f(x, y) = x^2 - xy^3 + y^5 - 17 = 0$ to put it in the proper form for the Implicit Function Theorem. Finally, we compute $\partial f/\partial y = -3xy^2 + 5y^4$ and evaluate at $(5, 2)$. We find $(\partial f/\partial y)(5, 2) = 20$, which is invertible. Thus y is defined as a function of x near $(x, y) = (5, 2)$.

b) Then, estimate the y value which corresponds to $x = 4.8$.

Now $\partial f/\partial x = 2x - y^3$. Evaluating at $(5, 2)$, we find $(\partial f/\partial x)(5, 2) = 2$. The Implicit function Theorem then implies $(dy/dx)(5) = -2/20 = -1/10$. Then since $\Delta x = -0.2$, $\Delta y = (dy/dx)(\Delta x) = 0.02$, so $y = 2.02$.

15.6 Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

a) If $x_1 = 6$ and $x_2 = 3$, find a y which satisfies $F(x_1, x_2, y) = 0$.

The only real solution is $y = -3$.

b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1 = 6$ and $x_2 = 3$?

Since $d_yF(6, 3) = 3(-3)^2 = 27$ is invertible, the Implicit Function Theorem implies that y is a function of (x_1, x_2) near $(6, 3)$.

c) If so, compute $(\partial y/\partial x_1)(6, 3)$ and $(\partial y/\partial x_2)(6, 3)$.

A short calculation shows $(\partial y/\partial x_1)(6, 3) = -4/9$ and $(\partial y/\partial x_2)(6, 3) = 2/9$.

d) If x_1 increases to 6.2 and x_2 decreases to 2.9, estimate the corresponding change in y.

The answer is $\Delta y = -(4/9)\Delta x_1 + (2/9)\Delta x_2 = -(4/9)(.2) - (2/9)(.1) = -1/9$.

15.13 A firm uses x hours of unskilled labor and y hours of skilled labor each day to produce $Q(x, y) = 60x^{2/3}y^{1/3}$ units of output per day. It currently employs 64 hours of unskilled labor and 27 hours of skilled labor per day.

a) What is its current output?

The current output is $60(16)(3) = 2880$.

b) In what direction (expressed as a unit vector) should it change (x, y) if it wants to increase output most rapidly?

We compute $dQ = (40x^{-1/3}y1/3, 20x^{2/3}y{-}2/3)^T$. Evaluating at $(64, 27)$, we find $dQ(64, 27) = (30, 320/9)^T$. We normalize this to find the direction of fastest increase, which is approximately $(0.645, 0.764)^T$.

c) The firm is planning to hire an additional hour and one-half of skilled labor. Use calculus to estimate the corresponding change in unskilled labor that would keep its output at the current level.

We consider the equation $60x^{2/3}y^{1/3} = 2880$, or $f(x, y) = x^{2/3}y^{1/3} - 48 = 0$. This implicitly defines y as a function of x. Then $dy/dx = -(\partial f/\partial x)/(\partial f/\partial y) = -2y/x$, which we evaluate at $(64, 27)$, obtaining -0.84. Then $1.5 = (dy/dx)(\Delta x)$, so $\Delta x = 1.78$.