Homework Assignment #5

12.25 Show that the three examples in the last paragraph are neither open nor closed.

The three examples are $(a, b]$ in $\mathbb{R}^1$, the sequence $\{1/n : n = 1, 2, \dots\}$ in $\mathbb{R}^1$, and a line minus a point in R^2.

None of the examples are closed because each contains a sequence that converges to a limit outside the set. The sequence $a + (b - a)/2n$ is in $(a, b]$ and converges to $a \ni (a, b]$. The sequence $1/n$ converges to 0, which is not in the sequence. Any sequence in the line converging to the missing point does the trick in the third case.

None of the examples are open because in each case there is a point about which no balls are contained in the set. For any $\epsilon > 0$, $(b - \epsilon, b + \epsilon)$ contains points that are not in $(a, b]$. The interval $(2/3, 4/3)$ contains $1/1$, but no other points of the sequence $\{1/n\}$. Finally, a ball about any point of the line will contain points that are off to the side (recall that this line is in $\mathbb{R}^2$).

12.31 Which of the five sets in exercise 12.21 are compact sets?

None are compact. Sets (a) and (d) are not closed. Sets (b), (c), and (e) are not bounded.

13.11 Write the following linear functions in matrix form:

a) $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + 5x_3$,

b) $f(x_1, x_2) = (2x_1 - 3x_2, x_1 - 4x_2, x_1)^T$,

c) $f(x_1, x_2, x_3) = (x_1 - x_3, 2x_1 + 3x_2 - 6x_3, x_3 + 2x_2)^T$.

$$2x_1 - 3x_2 + 5x_3 = [2, -3, 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - 3x_2 \\ x_1 - 4x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_3 \\ 2x_1 + 3x_2 - 6x_3 \\ x_3 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -6 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

13.12 Write the following quadratic forms in matrix form:

a) $x_1^2 - 2x_1x_2 + x_2^2$,

b) $5x_1^2 - 10x_1x_2 - x_2^2$,

c) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$.

$$a)\ x_1^2 - 2x_1x_2 + x_2^2 = \mathbf{x}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$$

$$b)\ 5x_1^2 - 10x_1x_2 - x_2^2 = \mathbf{x}^T \begin{bmatrix} 5 & -5 \\ -5 & -1 \end{bmatrix} \mathbf{x}$$

$$c)\ x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3 = \mathbf{x}^T \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{bmatrix} \mathbf{x}$$

13.23 For each of the following functions, what is the domain and the image of f? Which ones are one-to-one? For those which are one-to-one, write the expression for the inverse. Which ones are onto?

$$a)\ f(x) = 3x - 7; \qquad b)\ f(x) = x^2 - 1; \qquad c)\ f(x) = e^x;$$
$$d)\ f(x) = x^3 - x; \qquad e)\ f(x) = x/(x^2+1); \qquad f)\ f(x) = x^3;$$
$$g)\ f(x) = 1/x; \qquad h)\ f(x) = \sqrt{x-1}; \qquad i)\ f(x) = xe^{-x}.$$

In the following, I presume that "onto" refers to the target space. If you are stuck, try to compute the inverse by setting $y = f(x)$ and solving for x. If that is not feasible, try to graph the function. $\mathbb{R}^1$.

a) Domain: $\mathbb{R}$; image $\mathbb{R}$; one-to-one; inverse $g(y) = (y+7)/3$; onto.

b) Domain: $\mathbb{R}$; image $[-1, \infty)$; not one-to-one; not onto.

c) Domain: $\mathbb{R}$; image $(0, \infty)$; one-to-one; inverse $g(y) = \ln y$; not onto.

d) Domain: $\mathbb{R}$; image $\mathbb{R}$; not one-to-one; onto.

e) Domain: $\mathbb{R}$; image $[-1/2, +1/2]$; not one-to-one

f) Domain: $\mathbb{R}$; image $\mathbb{R}$; one-to-one; inverse $g(y) = y^{1/3}$; onto.

g) Domain: $\mathbb{R} \setminus \{0\}$; image $\mathbb{R} \setminus \{0\}$; inverse $g(y) = 1/y$; not onto.

h) Domain: $[1, \infty)$; image $[0, \infty)$; one-to-one; inverse $g(y) = 1 + y^2$; not onto.

i) Domain: $\mathbb{R}$; image $(\infty, e^{-1}]$; not one-to-one, not onto