Homework Assignment #4

12.2 Explain why each of the following sets is not a subsequence of

$$\left\{\frac{1}{1}, \frac{3}{1}, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{3}{3}, \frac{1}{4}, \cdots\right\}:$$

$$a) \left\{\frac{1}{1}, \frac{3}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}, \qquad b) \left\{\frac{3}{1}, \frac{3}{2}, \frac{3}{3}\right\}, \qquad c) \left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}.$$

The original sequence has a numerator alternating between 1 and 3, while the denominator starts at 1 and increases by 1 at every odd term. Sequence (a) is not a subsequence because does not follow the same order as the original sequence where $\frac{1}{2}$ precedes $\frac{3}{2}$. Alternative (b) is not a subsequence because it is not an infinite sequence. Sequence (c) is not a subsequence because it includes a term, $\frac{2}{1}$, that is not in the original sequence.

12.7 Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers that convergest to x_0 and that all x_n and x_0 are nonzero.

a) Prove that there is a positive number B such that $|x_n| \geq B$ for all n.

b) Using a, prove that $\{1/x_n\}$ converges to $1/x_0$.

a) Since $x_n \to x_0$, we may choose N so that $|x_n - x_0| < |x_0|/2$ for $n \geq N$. Then $|x_0| \leq |x_n| + |x_0 - x_n| \leq |x_n| + |x_0|/2$. Subtracting $|x_0|/2$ from both ends yields $|x_n| > |x_0|/2$ for $n \geq N$. Now let $B = min\{|x_1|/2, |x_2|/2, \ldots, |x_{N-1}|/2, |x_0|/2\}$. Since each x_n and x_0 are nonzero, $B > 0$. For $n < N$, $|x_n| \geq 2B > B$ by construction while for $n \geq N$, $|x_n| > |x_0|/2 \geq B$ by our original choice of N. This establishes the result.

b) Let $\epsilon > 0$ be arbitrary. Choose N such that $|x_n - x_0| < B|x_0|\epsilon$. Then, for $n \geq N$,

$$\begin{aligned} |x_n - x_0| &= \frac{1}{|x_n|\,|x_0|}|x_n - x_0| \\ &< \frac{1}{|x_n|\,|x_0|}B|x_0|\epsilon \\ &= \frac{B}{|x_n|}\epsilon < \epsilon \end{aligned}$$

where the last inequality follows from part (a). This establishes that $1/x_n \to 1/x_0$.

12.14 Prove that the strictly positive orthant $\mathbb{R}^m_{++} \equiv \{(x_1, \ldots, x_m) : x_i > 0 \text{ for } i = 1, \ldots, m\}$ is an open subset of $\mathbb{R}^m_{++}$ by finding a formula for ϵ in terms of the x_i's.

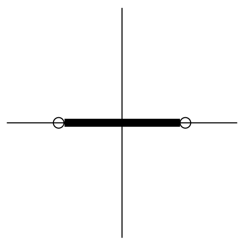
Let $x \in \mathbb{R}^m_{++}$. Let $\epsilon = \min\{x_1, \ldots, x_m\}$. Let $y \in B_\epsilon(x)$. Then $|y_i - x_i| \leq \|y - x\| < \epsilon$. Thus $-\epsilon < y_i - x_i < \epsilon$, which implies $x_i - \epsilon < y_i$. But $x_i - \epsilon \geq 0$, so $y_i > 0$. Since this holds for every $i = 1, \ldots, m$, $y \in \mathbb{R}^m_{++}$.

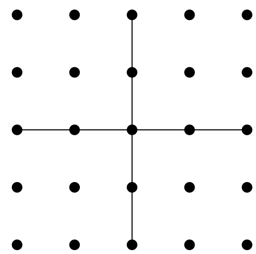
12.18 Show that *closed intervals* in $\Re^1$ — sets of the form $\{x : a \leq x \leq b\}$ for fixed numbers a and b — are closed sets.

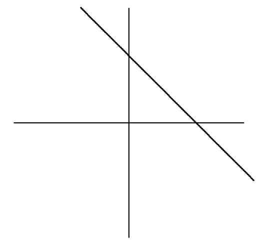
Let $[a, b]$ be a closed interval and $x_n \in [a, b]$ with $x_n \to x$. We must show $x \in [a, b]$. By Theorem 12.4, $a \leq x \leq b$, so $x \in [a, b]$. Therefore $[a, b]$ is closed.

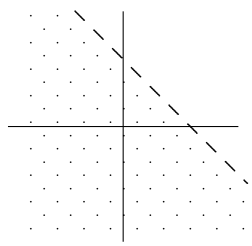
12.21 For each of the following subsets of the plane, draw the set, state whether it is open, closed, or neither, and justify your answer in a word or two:

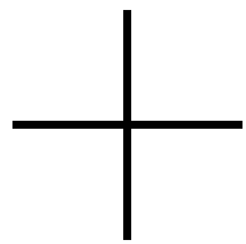
$$a)\ \{(x, y) : -1 < x < +1, y = 0\}, \qquad b)\ \{(x, y) : x \text{ and } y \text{ are integers}\},$$
$$c)\ \{(x, y) : x + y = 1\}, \qquad d)\ \{(x, y) : x + y < 1\}, \qquad e)\ \{(x, y) : x = 0 \text{ or } y = 0\}.$$


Set (a)


Set (b)


Set (c)


Set (d)


Set (e)

a) This set is neither open ($B_\epsilon(0, 0)$ pokes out), nor closed (the limit points $(-1, 0)$ and $(1, 0)$ are not included).

b) This set is closed. Any convergent sequence with integral coordinates must eventually be constant since there is only one integer point within any distance $\epsilon < 1/2$

c) This set is not open ($B_\epsilon(1, 0)$ pokes out). It is closed (if $f(x, y) = x + y$, f is continuous and the set is $f^{-1}(\{1\})$.).

d) This set is open (using the previous f, it is $f^{-1}(-\infty, 0)$). It is not closed since $(1, -1/n)$ is in the set and converges to a point outside the set.

e) This set is not open ($B_\epsilon(0, 0)$ pokes out), but is closed as the union of two closed sets (the coordinate axes).