Homework Assignment #2

8.7: Show that $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ are idempotent.

We simply compute the square of each matrix, obtaining the original back again.

8.17 Suppose that $a = 0$ but $c \neq 0$ in (5). Show that one obtains the same inverse (7) for A.

We proceed under the assumption that $0 \neq ad - bc = -bc$, which is equivalent to assuming that neither b nor c is zero. Otherwise, the matrix is not invertible.

Form the augmented matrix and row-reduce:

$$\begin{pmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \to \begin{pmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{-d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix}$$

This yields inverse:

$$\begin{pmatrix} \frac{-d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{pmatrix} = \frac{1}{-bc} \begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix},$$

which matches (7).

8.24

a) Use Theorem 8.8 to prove that a 2×2 lower- or upper- triangular matrix is invertible if and only if each diagonal entry is nonzero.

b) Show that the inverse of a 2×2 lower triangular matrix is lower triangular.

c) Show that the inverse of a 2×2 upper triangular matrix is upper triangular.

Theorem 8.8 tells us that a 2×2 matrix in the form (5) is invertible if and only if $ad - bc \neq 0$, in which case the inverse is (7). For part (a), note that an upper triangular matrix has $c = 0$ while a lower triangular matrix has $b = 0$. Either way, $bc = 0$, so $ad - bc = ad$. By Theorem 8.8, such a matrix is invertible if and only if $ad \neq 0$, which is equivalent to $a \neq 0$ and $d \neq 0$.

For parts (b) and (c), note that the inverse of a (upper or lower) triangular matrix is

$$A^{-1} = \frac{1}{ad} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

When $b = 0$ (lower triangular), A^{-1} is also lower triangular by the inverse formula and when $c = 0$ (upper triangular), A^{-1} is also upper triangular by the inverse formula.

8.28: What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}?$$

If any of the d_i are zero, the matrix is not invertible. If none of the d_i are zero, the inverse is

$$D = \begin{pmatrix} 1/d_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1/d_n \end{pmatrix}.$$

8.29 Show that the inverse of a 2×2 symmetric matrix S is symmetric.

We use Theorem 8.8 which tells us that this only makes sense when $ad - bc \neq 0$. In that case, use the fact that $b = c$ and formula (7) to write:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

Since the off-diagonal elements are equal, A^{-1} is symmetric.

9.3: Compute the expression on the right-hand side of (5). Show that it equals the expression calculated in Exercise 9.1.

The right-hand side of (5) is $a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$. Now $C_{12} = -a_{21}a_{33} + a_{23}a_{31}$, $C_{23} = a_{11}a_{33} - a_{13}a_{31}$, and $C_{32} = -a_{11}a_{23} + a_{21}a_{13}$. Thus the right-hand side of (5) becomes

$$-a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13},$$

which should match the six-term expression in Exercise 9.1.