Mathematical Economics Midterm #2, November 8, 2001

1. Consider the sequence $x_n = 2^n + (-2)^n + 1/n^2$.

   a) Does the sequence converge? If so, what is its limit?

   The sequence does not converge. In fact, $|x_n - x_{n+1}| > 2^n$, so the terms get farther apart.

   b) If the sequence doesn't converge, does it have any convergent subsequences? If so, identify one of them and compute its limit.

   Yes, it has convergent subsequences. One such subsequence is the subsequence of odd terms, $x_{n_j} = x_{2j+1} = 1/(2j+1)^2$. This subsequence converges to 0.

2. Find all local and global maxima and minima of the function $f(x,y) = (2/3)x^3 + x^2 + 2y^2 - 2xy + 4x - 10y$.

   First compute $df = (2x^2 + 2x - 2y + 4, 4y - 2x - 10)$. Setting $df = (0,0)$, we find $2y = x + 5$ from the second equation. Substituting in the first equation, this implies $2x^2 + x - 1 = 0$. This has solutions $x = -1$ and $x = 1/2$. It follows that the critical points are $(-1, 2)$ and $(1/2, 11/4)$.

   We next compute the Hessian:

$$d^2 f = \begin{bmatrix} 4x+2 & -2 \\ -2 & 4 \end{bmatrix}.$$

   At $(-1,2)$, we obtain $H_1 = -2 < 0$ and $H_2 = -8 - 4 = -12 < 0$. The Hessian is indefinite and $(-1,2)$ is neither a local minimum nor local maximum. At $(1/2, 11/4)$, we obtain $H_1 = 4 > 0$ and $H_2 = 16 - 4 = 12 > 0$. The Hessian is now positive definite and $(1/2, 11/4)$ is a local minimum.

   There is no global maximum. In fact $\lim_{x\to+\infty} f(x,0) = +\infty$. There is also no global minimum since $\lim_{x\to-\infty} f(x,0) = -\infty$

3. Consider the quadratic form $Q(x,y) = x^2 + 8xy + y^2$.

   a) Does the quadratic form have a maximum or minimum?

   This quadratic form has associated symmetric matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}.$$

   The first leading principal minor is $1 > 0$, while the second leading principal minor is $1 - 16 = -15 < 0$. This violates the sign patterns for positive or negative definite, so the matrix is indefinite. The form has neither maximum nor minimum.

b) Now impose the constraint $2x+3y=0$. Does the quadratic form have a constrained maximum or minimum?

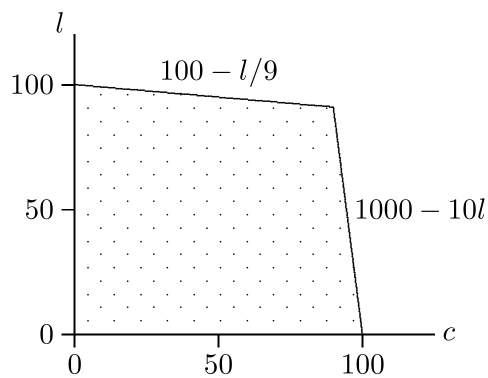
In this case we consider the bordered matrix:

$$H = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}.$$

The only princial minor to consider is the determinant of the whole matrix, which is $35 > 0$. Since $n = 2$ and so $(-1)^n = 1$, we have $(-1)^n \det H = 1$. This implies the form attains a maximum at $(0,0)$.

4. Suppose that utility is a continuous function from $\mathbb{R}^2_+$ to $\mathbb{R}$. The budget set is $\{(c,l) : (c,l) \geq \mathbf{0}, c \leq 1000 - 10l, c \leq 100 - l/9\}$.

a) Draw the budget set.



Budget Set

b) Does the consumer's problem have a solution? That is, can utility be maximized over the budget set? Explain.

Yes, the consumer's problem has a solution. It suffices to show the budget set is compact since the Weierstrass Theorem will imply there is a solution.

The budget set is the interesection of four sets: $\{(c,l) : l \geq 0\}$, $\{(c,l) : c \geq 0\}$, $\{(c,l) : c \leq 1000-10l\}$, and $\{(c,l) : c \leq 100-l/9\}$. Each of these sets is the inverse image of a closed interval under a continuous function, and so each is closed. As the intersection of closed sets, the budget set is closed.

Since $l \geq 0$, $c \leq 100 - l/9 \leq 100$. Since $c \geq 0$, $10l \leq 1000 - c \leq 1000$, so $l \leq 100$. The budget set is contained in $[0,100] \times [0,100]$ (see the diagram), and thus is bounded. Since it is both closed and bounded, it is compact. By the Weierstrass Theorem, the consumer's problem has a solution.

Note: Polygonal budget sets of this type arise when consumers face progressive taxation, where the number of sides depends on the number of tax brackets. The phase-out of welfare benefits as income increases has a similar effect.

5. Let $f(x,y,z) = x^2 + xy + y^2 + z^3 - 4$.

a) Find a point (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

There are many solutions: The point $(x_0, y_0, z_0) = (1,1,1)$ seemed to be the consensus choice, but $(0,2,0)$, $(2,0,0)$, $(5,1,-3)$, $(1,5,-3)$, $(-2,1,1)$, $(1,-2,1)$, and $(2,2,2)$ are other integral solutions. Of course, you are not restricted to integers, but they often make calculation easier.

b) Can x be expressed as a function $g(y,z)$ in some neighborhood of (x_0, y_0, z_0)?

We use the point $(x_0, y_0, z_0) = (1,1,1)$. Here $\partial f/\partial x = 2x_0 + y_0 = 3 \neq 0$. The Implicit Function Theorem now implies that we can write $x = g(y,z)$ in some neighborhood of $(1,1,1)$.

Note: In this case, we can use the quadratic formula to find an expression for g,

$$g(y,z) = \frac{-y + \sqrt{16 - 3y^2 - 4z^3}}{2}.$$

Of course, this is only valid for $16 - 3y^2 - 4z^3 \geq 0$. Notice that the positive square root is chosen so that $g(y_0, z_0) = g(1,1) = 1 = x_0$. Had we used $(-2,1,1)$ as our reference point, we would use the negative square root.

c) Compute $dg(y_0, z_0)$.

By the Implicit Function Theorem,

$$\begin{aligned} dg(y_0, z_0) &= -\left(\frac{\partial f}{\partial x}\right)^{-1} \begin{bmatrix} \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} \\ &= \frac{-1}{2x_0 + y_0} \begin{bmatrix} x_0 + 2y_0 \\ 3z_0 \end{bmatrix} \\ &= -\frac{1}{3} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \end{aligned}$$

Notice that this is an easier computation than using the formula for g given in part (b).